

MOMENT FORMULAS FOR THE QUASI-NILPOTENT DT-OPERATOR

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ABSTRACT. Let T be the quasi-nilpotent DT-operator. By use of Voiculescu's amalgamated R -transform we compute the moments of $(T - \lambda 1)^*(T - \lambda 1)$ where $\lambda \in \mathbb{C}$, and the Brown-measure of $T + \sqrt{\epsilon}Y$, where Y is a circular element $*$ -free from T for $\epsilon > 0$. Moreover we give a new proof of Śniady's formula for the moments $\tau(((T^*)^k T^k)^n)$ for $k, n \in \mathbb{N}$.

1. INTRODUCTION

The quasi-nilpotent DT-operator T was introduced by Dykema and the second author in [DH1]. It can be described as the limit in $*$ -moments for $n \rightarrow \infty$, of random matrices of the form

$$T^{(n)} = \begin{pmatrix} 0 & t_{1,2} & \cdots & t_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{n-1,n} \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\{\Re(t_{ij}), \Im(t_{ij})\}_{1 \leq i < j \leq n}$ is a set of $n(n-1)$ independent identically distributed Gaussian random variables with mean 0 and variance $\frac{1}{2n}$. More precisely, T is an element in a finite von Neumann algebra, M , with a faithful normal tracial state, τ , such that for all $s_1, s_2, \dots, s_k \in \{1, *\}$,

$$(1.1) \quad \tau(T^{s_1} T^{s_2} \dots T^{s_k}) = \lim_{n \rightarrow \infty} \mathbb{E}[\mathrm{tr}_n((T^{(n)})^{s_1} (T^{(n)})^{s_2} \dots (T^{(n)})^{s_k})],$$

where tr_n is the normalized trace on $M_n(\mathbb{C})$. Moreover the pair $(T, W^*(T))$ is uniquely determined up to $*$ -isomorphism by (1.1). The quasi-nilpotent DT-operator can be realized as an element in the free group factor, $L(\mathbb{F}_2)$, in the following way (cf. [DH1, Sect. 4]): Let (D_0, X) be a pair of free selfadjoint elements in a tracial W^* -probability space (M, τ) , such that $d\mu_{D_0}(t) = 1_{[0,1]}(t)dt$ and X is semi-circular

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distributed, i.e. $d\mu_X(t) = \frac{1}{2\pi}\sqrt{4-t^2}1_{[-2,2]}(t)dt$. Then $W^*(D_0, X) \simeq W^*(D_0) \star W^*(X) \simeq L(\mathbb{F}_2)$. Put

$$T_N = \sum_{j=1}^{2^N} p_{N,j} X q_{N,j}$$

for $N = 1, 2, \dots$, where

$$p_{N,j} = 1_{[\frac{j-1}{2^N}, \frac{j}{2^N}]}(D_0), \quad q_{N,j} = 1_{[\frac{j}{2^N}, 1]}(D_0),$$

for $j = 1, 2, \dots, 2^N$. Then $(T_N)_{N=1}^\infty$ converges in norm to an operator $T \in W^*(D_0, X)$, and the $*$ -moments of T are given by (1.1), i.e. T is a realization of the quasi-nilpotent DT-operator. In the notation of [DH1, Sect. 4], $T = \mathcal{UT}(X, \lambda)$, where $\lambda : L^\infty[0, 1] \rightarrow W^*(D_0)$ is the $*$ -isomorphism given by $\lambda(f) = f(D_0)$ for $f \in L^\infty([0, 1])$. In the following we put $\mathcal{D} = W^*(D_0) \simeq L^\infty([0, 1])$ and let $E_{\mathcal{D}}$ denote the trace-preserving conditional expectation of $W^*(D_0, X)$ onto \mathcal{D} .

In this paper we apply Voiculescu's \mathcal{R} -transform with amalgamation to compute various $*$ -moments of T and of operators closely related to T . First we compute in section 3 moments and the scalar valued \mathcal{R} -transform of $(T - \lambda 1)^*(T - \lambda 1)$ for $\lambda \in \mathbb{C}$. The specialized case of $\lambda = 0$ was treated in [DH1] by more complicated methods. In section 4 we consider the operator

$$T + \sqrt{\epsilon}Y,$$

where Y is a circular operator $*$ -free from T and $\epsilon > 0$. By random matrix considerations it is easily seen, that if T_1 and T_2 are two quasi-nilpotent DT-operators, which are $*$ -free with respect to amalgamation over the same diagonal, \mathcal{D} , then $T + \sqrt{\epsilon}Y$ has the same $*$ -distribution as $S = \sqrt{a}T_1 + \sqrt{b}T_2$, when $a = 1 + \epsilon$ and $b = \epsilon$ (cf. [Aa1]). We use this fact to prove, that the Brown measure of $T + \sqrt{\epsilon}Y$ is equal to the uniform distribution on the closed disc $\overline{B}(0, \log(1 + \frac{1}{\epsilon})^{-\frac{1}{2}})$ in the complex plane. Moreover we show, that the spectrum of $T + \sqrt{\epsilon}Y$ is equal to this disc, and that $T + \sqrt{\epsilon}Y$ is not a DT-operator for any $\epsilon > 0$.

In [DH1] it was conjectured, that

$$(1.2) \quad \tau(((T^*)^k T^k)^n) = \frac{n^{nk}}{(nk+1)!}$$

for $n, k \in \mathbb{N}$. This formula was proved by Śniady in [Śn2]. Śniady's proof of (1.2) is based on Speicher's combinatorial approach to free probability with amalgamation from [Sp1]. The key step in the proof

of (1.2) was to establish a recursion formula for the \mathcal{D} -valued moments,

$$(1.3) \quad E_{\mathcal{D}} \left(((T^*)^k T^k)^n \right)$$

for each fixed $k \in \mathbb{N}$. Śniady's recursion formula for the \mathcal{D} -valued moments (1.3), was later used by Dykema and the second author to prove, that

$$W^*(T) = W^*(D_0, X) \simeq L(\mathbb{F}_2)$$

and that T admits a one parameter family of non-trivial hyperinvariant subspaces (cf. [DH2]). In section 5 and section 6 of this paper we give a new proof of Śniady's recursion formula for the \mathcal{D} -valued moments (1.3), which at the same time gives a new proof of (1.2). The new proof is based on Voiculescu's \mathcal{R} -transform with respect to amalgamation over $M_{2k}(\mathcal{D})$, the algebra of $2k \times 2k$ matrices over \mathcal{D} .

2. PRELIMINARIES

In this section we give a few preliminaries on amalgamated probability theory. Let \mathcal{A} be a unital Banach algebra, and let \mathcal{B} be a Banach-sub-algebra containing the unit of \mathcal{A} . Then a map, $E_{\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{B}$, is a conditional expectation if

- (a) $E_{\mathcal{B}}$ is linear,
- (b) $E_{\mathcal{B}}$ preserves the unit i.e. $E_{\mathcal{B}}(1) = 1$
- (c) and $E_{\mathcal{B}}$ has the \mathcal{B} , \mathcal{B} bi-module property i.e. $E_{\mathcal{B}}(b_1 a b_2) = b_1 a b_2$ for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.

If \mathcal{B} , \mathcal{A} and $E_{\mathcal{B}}$ are as above we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a \mathcal{B} -probability space. If $\phi : \mathcal{A} \rightarrow \mathbb{C}$ is a state on \mathcal{A} which respects $E_{\mathcal{B}}$, i.e. $\tau = \tau \circ E_{\mathcal{B}}$, we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is compatible to the (non-amalgamated) free probability space (\mathcal{A}, ϕ) .

If $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a \mathcal{B} -probability space and $a \in \mathcal{A}$ is a fixed variable, we define the amalgamated Cauchy transform of a by

$$G_a(b) = E_{\mathcal{B}}((b - a)^{-1}).$$

for $b \in \mathcal{B}$ and $b - a \in \mathcal{B}_{\text{inv}}$. The Cauchy transform is 1-1 in $\{b \in \mathcal{B}_{\text{inv}} \mid \|b^{-1}\| < \epsilon\}$ for ϵ sufficiently small and Voiculescu's amalgamated \mathcal{R} -transform [Voi] is now defined for $a \in \mathcal{A}$ by

$$(2.1) \quad \mathcal{R}_a(b) = G_a^{(-1)}(b) - b^{-1},$$

for b being an invertible element of \mathcal{B} suitably close to zero. It turns out that this definition coincides on invertible element with Speicher's

definition of the amalgamated \mathcal{R} -transform (cf. [Sp1, Th. 4.1.2] and [Aa2]);

$$(2.2) \quad \mathcal{R}_a(b) = \sum_{n=1}^{\infty} \kappa_n^{\mathcal{B}}(a \otimes_{\mathcal{B}} ba \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} ba).$$

We will need the following useful lemma for solving equations involving the amalgamated R -transform and Cauchy-transform.

Lemma 2.1. *Let $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ be a \mathcal{B} -probability space, and let $a \in \mathcal{A}$. Then there exists $\delta > 0$ such that if $b \in \mathcal{B}$ is invertible, $\|b\| < \delta$, $|\mu| > \frac{1}{\delta}$ and*

$$\mathcal{R}_a^{\mathcal{B}}(b) + b^{-1} = \mu 1_{\mathcal{A}}$$

then $b = G_a^{\mathcal{B}}(\mu 1_{\mathcal{A}})$.

Proof. Let $\delta = \frac{1}{11\|a\|}$ and define $g_b(b) = G_a^{\mathcal{B}}(b^{-1})$. By [Aa2, Prop. 2.3] we know that g_a maps $\mathcal{B}(0, \frac{1}{4\|a\|})$ bijectively onto a neighborhood of zero containing $\mathcal{B}(0, \frac{1}{11\|a\|})$ and furthermore that

$$g_a^{\langle -1 \rangle} \left(\mathcal{B}(0, \frac{1}{11\|a\|})_{\text{inv}} \right) \subseteq \mathcal{B}(0, \frac{2}{11\|a\|})_{\text{inv}}.$$

By definition we know that

$$\mathcal{R}_a^{\mathcal{B}}(b) = G_a^{\mathcal{B}^{\langle -1 \rangle}}(b) + b^{-1} = (g_a^{\langle -1 \rangle}(b))^{-1} + b^{-1}$$

so if $\mathcal{R}_a(b) + b^{-1} = \mu 1_{\mathcal{A}}$ then

$$\mu 1_{\mathcal{A}} = g_a^{\langle -1 \rangle}(b) - b^{-1} + b^{-1} = (g_a^{\langle -1 \rangle}(b))^{-1}$$

and thus

$$(2.3) \quad g_a^{\langle -1 \rangle}(b) = \frac{1}{\mu} 1_{\mathcal{A}}.$$

If $|\mu| > \frac{1}{\delta}$ then especially $\frac{1}{|\mu|} < \frac{1}{4\|a\|}$ so $\frac{1}{\mu} 1_{\mathcal{A}}$ is in the bijective domain of g_a , so applying g_a on both sides of (2.3) we get exactly

$$G_a^{\mathcal{B}}(\mu 1_{\mathcal{A}}) = g_a(\frac{1}{\mu} 1_{\mathcal{A}}) = b$$

since also $\|b\| < \frac{1}{11\|a\|}$. □

If $a \in \mathcal{A}$ is a random variable in the \mathcal{B} -probability space $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$, then following Speicher we define a to be \mathcal{B} -Gaussian [Sp1, Def 4.2.3] if only \mathcal{B} -cumulants of length 2 survive. From (2.2) it follows that in this case the \mathcal{R} -transform has a particularly simple form, namely,

$$(2.4) \quad \mathcal{R}_a(b) = \kappa_2^{\mathcal{B}}(a \otimes_{\mathcal{B}} ba) = E_{\mathcal{B}}(aba).$$

In the following theorem (which is probably not a new one we just could not find a proper reference) concerning cumulants we have adopted the notation of Speicher from [Sp1].

Lemma 2.2. *Let $N \in \mathbb{N}$ and let $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ be a \mathcal{B} -probability space. Then $(M_N(\mathcal{B}) \subset M_N(\mathcal{A}), E_{M_N(\mathcal{B})})$ is a $M_N(\mathcal{B})$ -probability space with cumulants determined by the following formula:*

$$\begin{aligned} \kappa_n^{M_N(\mathcal{B})}((m_1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (m_n \otimes a_n)) \\ = (m_1 \cdots m_n) \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n) \end{aligned}$$

when $m_1, \dots, m_n \in M_N(\mathbb{C})$ and $a_1, \dots, a_n \in \mathcal{A}$.

We have of course made the identification $M_N(\mathcal{A}) \cong M_N(\mathbb{C}) \otimes \mathcal{A}$.

Proof. Since $M_N(\mathbb{C}) \subset M_N(\mathcal{B})$ we observe that

$$\begin{aligned} \kappa_n^{M_N(\mathcal{B})}((m_1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (m_n \otimes a_n)) \\ = ((m_1 \cdots m_n) \otimes 1) \cdot \\ \kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)). \end{aligned}$$

To finish the proof we claim that

$$\begin{aligned} (2.5) \quad \kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)) = \\ 1 \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n). \end{aligned}$$

The case $n = 1$ is obvious since

$$1_N \otimes \kappa_1^{\mathcal{B}}(a_1) = 1_N \otimes E_{\mathcal{B}}(a_1) = E_{M_N(\mathcal{B})}(1 \otimes a_1) = \kappa_1^{M_N(\mathcal{B})}(1 \otimes a_1).$$

Now assume that the claim is true for $1, 2, \dots, n-1$. Then (2.5) has an obvious extension to noncrossing partions of length less than or equal to $n-1$. Hence

$$\begin{aligned} 1_N \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n) \\ = 1_N \otimes E_{\mathcal{B}}(a_1 \cdots a_n) - \sum_{\pi \in NC(n), \pi \neq 1_n} 1 \otimes \kappa_{\pi}^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n) \\ = E_{M_N(\mathcal{B})}(1 \otimes_{M_N(\mathcal{B})} a_1 \cdots a_n) \\ - \sum_{\pi \in NC(n), \pi \neq 1_n} \kappa_{\pi}^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)) \\ = \kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)). \end{aligned}$$

By induction this proves the lemma. \square

Assume that \mathcal{M} contains a pair (D_0, X) of τ -free selfadjoint elements such that $d\mu_{D_0}(t) = 1_{[0,1]}(t)dt$ and X is a semicircular distributed. Put $\mathcal{D} = W^*(D_0)$. Then $\lambda : L^\infty([0, 1]) \rightarrow \mathcal{D}$ given by

$$\lambda(f) = f(D_0),$$

for $f \in L^\infty([0, 1])$ is a $*$ -isomorphism of $L^\infty([0, 1])$ onto \mathcal{D} and

$$\tau \circ \lambda(f) = \int_0^1 f(t)dt, \quad f \in L^\infty([0, 1]).$$

We will identify \mathcal{D} with $L^\infty([0, 1])$ and thus consider elements of \mathcal{D} as functions. As explained in the introduction, we can realize the quasi-nilpotent DT-operator as the operator $T = \mathcal{U}\mathcal{T}(X, \lambda)$ in $W^*(D_0, X) \simeq L(\mathbb{F}_2)$.

Define for $f \in \mathcal{D} \simeq L^\infty([0, 1])$

$$(2.6) \quad (L^*(f))(x) := \int_0^x f(t)dt \quad \text{and} \quad (L(f))(x) := \int_x^1 f(t)dt.$$

From the appendix of [DH2] it follows that (T, T^*) is a \mathcal{D} -Gaussian pair and that the covariances of (T, T^*) are given by the following lemma

Lemma 2.3. [DH2, Appendix] *Let $f \in \mathcal{D}$. Then*

$$E_{\mathcal{D}}(TfT^*) = L(f) \quad \text{and} \quad E_{\mathcal{D}}(T^*fT) = L^*(f)$$

and $E_{\mathcal{D}}(TfT) = E_{\mathcal{D}}(T^*fT^*) = 0$.

3. MOMENTS AND \mathcal{R} -TRANSFORM OF $(T - \lambda 1)^*(T - \lambda 1)$

Let T be the quasi-diagonal DT-operator and define

$$\tilde{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

Since (T, T^*) is a \mathcal{D} -Gaussian pair, it follows from lemma 2.2, that cumulants of the form

$$\kappa_n^{M_2(\mathcal{D})}((m_1 \otimes a_1) \otimes_{M_2(\mathcal{D})} \cdots \otimes_{M_2(\mathcal{D})} (m_n \otimes a_n))$$

vanishes when $n \neq 2$, $m_1, m_2, \dots, m_n \in M_2(\mathbb{C})$ and $a_1, a_2, \dots, a_n \in \{T, T^*\}$. Hence by the linearity of $\kappa_n^{M_2(\mathcal{D})}$,

$$\kappa_n^{M_2(\mathcal{D})}(\tilde{T} \otimes_{M_2(\mathcal{D})} \tilde{T} \otimes_{M_2(\mathcal{D})} \cdots \otimes_{M_2(\mathcal{D})} \tilde{T}) = 0$$

when $n \neq 2$, i.e. \tilde{T} is a $M_2(\mathcal{D})$ -Gaussian element in $M_2(\mathcal{M})$ under the conditional expectation $E_{M_2(\mathcal{D})} : M_2(\mathcal{M}) \rightarrow M_2(\mathcal{D})$ given by

$$E_{M_2(\mathcal{D})} : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} E_{\mathcal{D}}(a_{11}) & E_{\mathcal{D}}(a_{12}) \\ E_{\mathcal{D}}(a_{21}) & E_{\mathcal{D}}(a_{22}) \end{pmatrix}.$$

Since \tilde{T} is $M_2(\mathcal{D})$ -Gaussian the \mathcal{R} -transform of \tilde{T} is by (2.4) the linear mapping $M_2(\mathcal{D}) \rightarrow M_2(\mathcal{D})$ given by

$$\begin{aligned} \mathcal{R}_{\tilde{T}}^{M_2(\mathcal{D})}(z) &= E_{M_2(\mathcal{D})}(\tilde{T}z\tilde{T}) \\ &= E_{M_2(\mathcal{D})} \left(\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \right) \\ &= E_{M_2(\mathcal{D})} \left(\begin{pmatrix} T^*z_{22}T & 0 \\ 0 & Tz_{11}T^* \end{pmatrix} \right) \\ &= \begin{pmatrix} E_{\mathcal{D}}(T^*z_{22}T) & 0 \\ 0 & E_{\mathcal{D}}(Tz_{11}T^*) \end{pmatrix} \\ &= \begin{pmatrix} L^*(z_{22}) & 0 \\ 0 & L(z_{11}) \end{pmatrix}. \end{aligned}$$

For $\lambda \in \mathbb{C}$, we put $T_\lambda - T_\lambda 1$ and define

$$\tilde{T}_\lambda = \begin{pmatrix} 0 & T_\lambda^* \\ T_\lambda & 0 \end{pmatrix} = \tilde{T} - \begin{pmatrix} 0 & \bar{\lambda}1 \\ \lambda 1 & 0 \end{pmatrix}$$

Since $\begin{pmatrix} 0 & \bar{\lambda}1 \\ \lambda 1 & 0 \end{pmatrix} \in M_2(\mathcal{D})$ we have by $M_2(\mathcal{D})$ -freeness that the \mathcal{R} -transform is additive [Sp1, Th. 4.1.22] i.e.

$$\mathcal{R}_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(z) = \mathcal{R}_{\tilde{T}}^{M_2(\mathcal{D})} - \begin{pmatrix} 0 & \bar{\lambda}1 \\ \lambda 1 & 0 \end{pmatrix} = \begin{pmatrix} L^*(z_{22}) & -\bar{\lambda}1 \\ -\lambda 1 & L(z_{11}) \end{pmatrix}.$$

One easily checks, that if $\delta \in \mathbb{C}$, $\delta \neq 0$, $\delta \neq -\frac{1}{|\lambda|^2}$ and $\mu \in \mathbb{C}$ is one of the two solutions to

$$\mu^2 = \frac{e^\sigma}{\sigma}(1 + |\lambda|^2\sigma),$$

then

$$(3.1) \quad \begin{cases} z_{11} = \mu\sigma e^{\sigma(x-1)} \\ z_{12} = -\bar{\lambda}\sigma \\ z_{21} = -\lambda\sigma \\ z_{22} = \mu\sigma e^{-\sigma x} \end{cases}$$

is a solution to

$$\mathcal{R}_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(z) + z^{-1} = \mu 1_2.$$

Here x is the variable for the function in $\mathcal{D} = L^\infty([0, 1])$. In particular z_{12} and z_{21} are constant operators. If $\sigma \rightarrow 0$ then $|\mu| \rightarrow \infty$ and $\|z\| \rightarrow 0$, so by lemma 2.1 there exists $\rho > 0$ such that $|\sigma| < \rho$ implies

$$G_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(\mu 1_2) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix},$$

where $(z_{ij})_{i,j \in \{1,2\}}$ is given by (3.1) and

$$\mu = \pm \sqrt{\frac{e^\sigma}{\sigma}(1 + |\lambda|^2\sigma)}.$$

On the other hand the Cauchy-transform of \tilde{T} in $\mu 1_2$ is

$$\begin{aligned} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} &= G_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(\mu 1_2) \\ &= E_{M_2(\mathcal{D})} \left(\left(\begin{pmatrix} \mu 1 & 0 \\ 0 & \mu 1 \end{pmatrix} - \begin{pmatrix} 0 & T_\lambda^* \\ T_\lambda & 0 \end{pmatrix} \right)^{-1} \right) \\ &= E_{M_2(\mathcal{D})} \left(\begin{pmatrix} \mu 1 & -T_\lambda^* \\ -T_\lambda & \mu 1 \end{pmatrix}^{-1} \right) \\ &= E_{M_2(\mathcal{D})} \left(\begin{pmatrix} \mu(\mu^2 1 - T_\lambda^* T_\lambda)^{-1} & T_\lambda^* (\mu^2 1 - T_\lambda T_\lambda^*)^{-1} \\ T_\lambda (\mu^2 1 - T_\lambda^* T_\lambda)^{-1} & \mu(\mu^2 1 - T_\lambda T_\lambda^*)^{-1} \end{pmatrix} \right). \end{aligned}$$

Thus

$$(3.2) \quad \begin{cases} z_{11} = \mu E_{\mathcal{D}}((\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) \\ z_{12} = E_{\mathcal{D}}(T_\lambda^* (\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) \\ z_{21} = E_{\mathcal{D}}(T_\lambda (\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) \\ z_{22} = \mu E_{\mathcal{D}}((\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) \end{cases}.$$

Combining (3.1) and (3.2) we have

$$(3.3) \quad \begin{cases} E_{\mathcal{D}}((\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) = \sigma e^{\sigma(x-1)} \\ E_{\mathcal{D}}(T_\lambda^* (\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) = -\bar{\lambda}\sigma \\ E_{\mathcal{D}}(T_\lambda (\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) = -\lambda\sigma \\ E_{\mathcal{D}}((\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) = \sigma e^{-\sigma x} \end{cases}.$$

We can now compute the \mathcal{R} -transform of $T_\lambda^* T_\lambda$ (wrt. \mathbb{C}) from (3.3) and the defining equality for μ^2 .

$$\begin{aligned} \text{tr} \left(\left(\frac{e^\sigma}{\sigma}(1 + |\lambda|^2\sigma) 1 - T_\lambda^* T_\lambda \right)^{-1} \right) &= \int_0^1 \sigma e^{\sigma(x-1)} dx \\ &= [e^{\sigma(x-1)}]_0^1 = 1 - e^{-\sigma}. \end{aligned}$$

Thus

$$G_{T_\lambda^* T_\lambda}^{\mathbb{C}} \left(\frac{e^\sigma}{\sigma}(1 + |\lambda|^2\sigma) \right) = 1 - e^{-\sigma}$$

i.e.

$$\mathcal{R}_{T_\lambda^* T_\lambda}^{\mathbb{C}}(1 - e^{-\sigma}) = \frac{e^\sigma}{\sigma}(1 + |\lambda|^2\sigma) - \frac{1}{1 - e^{-\sigma}}$$

for σ in a neighborhood of zero. Substituting $z = 1 - e^{-\sigma}$ we get $\sigma = -\log(1 - z)$, so

$$\mathcal{R}_{T_\lambda^* T_\lambda}^{\mathbb{C}}(z) = -\frac{1}{(1 - z) \log(1 - z)}(1 - |\lambda|^2 \log(1 - z)) - \frac{1}{z}.$$

Hence we have proved the following extension of [DH1, Theorem 8.7(b)]:

Theorem 3.1. *Let T be the quasinilpotent DT-operator. Let $\lambda \in \mathbb{C}$ and put $T_\lambda = T - \lambda 1$. Then*

$$\mathcal{R}_{T_\lambda^* T_\lambda}^{\mathbb{C}}(z) = -\frac{1}{(1 - z) \log(1 - z)} - \frac{1}{z} + \frac{|\lambda|^2}{1 - z}$$

for z in some neighborhood of 0.

We next determine the \mathcal{D} -valued (resp. \mathbb{C} -valued) moments of $T_\lambda^* T_\lambda$ for all $\lambda \in \mathbb{C}$. The special case $\lambda = 0$ was treated in [Śn2, Theorem 5] (resp. [DH1, Theorem 8.7(a)]) by different methods.

Theorem 3.2. *Let $\lambda \in \mathbb{C}$ and let T, T_λ be as in theorem 3.1*

(a) *Let Q_n be the sequence of polynomials on \mathbb{R} uniquely determined by the following recursion formula*

$$(3.4) \quad \begin{cases} Q_0(x) = 1, \\ Q_{n+1}(x) = |\lambda|^2 Q_n(x+1) + \int_0^x Q_n(y+1) dy \quad \text{for } n \geq 1. \end{cases}$$

Then

$$E_{\mathcal{D}}((T_\lambda^* T_\lambda)^n)(x) = Q_n(x), \quad x \in [0, 1], \quad n \in \mathbb{N}.$$

(b)

$$\tau((T_\lambda^* T_\lambda)^n) = \sum_{k=0}^n \frac{n^k}{(k+1)!} \binom{n}{k} |\lambda|^{2n-2k}, \quad n \in \mathbb{N}.$$

Proof. By (3.3), we have

$$(3.5) \quad E_{\mathcal{D}}\left(\left(\frac{e^\sigma}{\sigma}(1 + |\lambda|^2 \sigma)1 - T_\lambda^* T_\lambda\right)^{-1}\right) = \sigma e^{\sigma(x-1)}$$

for $\sigma \in B(0, \rho) \setminus \{0\}$ for some $\rho > 0$. Put

$$\psi(\sigma) = \frac{\sigma}{e^\sigma(1 + |\lambda|^2 \sigma)}, \quad \sigma \in \mathbb{C} \setminus \{-\frac{1}{|\lambda|^2}\}.$$

Since $\psi(0) = 0$ and $\psi'(0) = 1$, ψ has an analytic inverse $\psi^{(-1)}$ defined in a neighborhood $B(0, \delta)$ of 0, and we can choose $\delta > 0$, such that $\psi^{(-1)}(B(0, \delta)) \subset B(0, \rho)$. By (3.5)

$$E_{\mathcal{D}}\left(\left(\frac{1}{t}1 - T_\lambda^* T_\lambda\right)^{-1}\right) = \psi^{(-1)}(t)e^{\psi^{(-1)}(t)(x-1)}$$

for $t \in B(0, \delta) \setminus \{0\}$. By power series expansion of the left hand side, we get

$$(3.6) \quad \sum_{n=0}^{\infty} t^{n+1} E_{\mathcal{D}}((T_{\lambda}^* T_{\lambda})^n) = \psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)}$$

for $t \in B(0, \delta')$, where $0 < \delta' \leq \delta$ and where the LHS of (3.6) is absolutely convergent in the Banach space $L^{\infty}([0, 1])$. Hence by Cauchy's integral formulas

$$(3.7) \quad E_{\mathcal{D}}((T_{\lambda}^* T_{\lambda})^n) = \frac{1}{2\pi i} \int_C \frac{\psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)}}{t^{n+2}} dt$$

as a Banach space integral in $L^{\infty}([0, 1])$, where $C = \partial B(0, r)$ with positive orientation and $0 < r < \delta'$. For each fixed $x \in \mathbb{R}$

$$t \mapsto \psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)}$$

is an analytic function in $B(0, \delta')$ which is 0 for $t = 0$. Hence the function has a power series expansion of the form

$$(3.8) \quad \psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)} = \sum_{n=0}^{\infty} Q_n(x) t^{n+1}$$

for $t \in B(0, \delta')$, where the numbers $(Q_n(x))_{n=0}^{\infty}$ are given by

$$(3.9) \quad Q_n(x) = \frac{1}{2\pi i} \int_C \frac{\psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)}}{t^{n+2}} dt.$$

In particular the Q_n 's are continuous functions of $x \in \mathbb{R}$. Substituting $\sigma = \psi(t)$ in (3.8) we get

$$\sum_{n=0}^{\infty} Q_n(x) \psi(\sigma)^{n+1} = \sigma e^{\sigma(x-1)}$$

for $\sigma \in B(0, \rho')$, where $\rho' \in (0, \rho)$. Put

$$\begin{cases} R_0(x) = 0 \\ R_{n+1}(x) = |\lambda|^2 Q_n(x+1) + \int_0^x Q_n(x) dy, \quad n \geq 0. \end{cases}$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty} R_n(x) \psi(\sigma)^{n+1} &= \psi(\sigma) \left(1 + \sum_{n=0}^{\infty} R_{n+1}(x) \psi(\sigma)^{n+1} \right) \\
&= \psi(\sigma) \left(1 + |\lambda|^2 \left(\sum_{n=0}^{\infty} Q_n(x+1) \right) + \int_0^x \left(\sum_{n=0}^{\infty} Q_n(y+1) \right) dy \right) \\
&= \psi(\sigma) \left(1 + |\lambda|^2 \sigma e^{\sigma x} + \int_0^x \sigma e^{\sigma y} dy \right) \\
&= \psi(\sigma) (|\lambda|^2 \sigma + 1) e^{\sigma x} = \sigma e^{\sigma(x-1)} = \sum_{n=0}^{\infty} Q_n(x) \psi(\sigma)^{n+1}
\end{aligned}$$

for all $\sigma \in B(0, \rho')$. Since $\psi(B(0, \rho'))$ is an open neighborhood of 0 in \mathbb{C} , it follows that $R_n(x) = Q_n(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$.

Hence $(Q_n(x))_{n=0}^{\infty}$ is the sequence of polynomials given by the recursive formula (3.4). Moreover by (3.7) and (3.9), $E_{\mathcal{D}}((T_{\lambda}^* T_{\lambda})^n) = Q_n$ as functions in $L^{\infty}([0, 1])$. This proves (a).

(b) By (3.7), we have

$$\tau((T_{\lambda}^* T_{\lambda})^n) = \int_0^1 E_{\mathcal{D}}((T_{\lambda}^* T_{\lambda})^n) dx = \frac{1}{2\pi i} \int_C \frac{1 - e^{-\psi^{(-1)}(t)}}{t^{n+2}} dt.$$

Note that $C' = \psi(C)$ is a positively oriented simple path around 0. Hence by the substitution $t = \psi(\sigma)$, we get

$$\begin{aligned}
\tau((T_{\lambda}^* T_{\lambda})^n) &= \frac{1}{2\pi i} \int_{C'} \frac{\psi'(\sigma)}{\psi(\sigma)^{n+2}} (1 - e^{-\sigma}) d\sigma \\
&= \frac{1}{2\pi i} \int_{C'} \frac{1}{n+1} \frac{1}{\psi(\sigma)^{n+1}} \frac{d}{d\sigma} (1 - e^{-\sigma}) d\sigma \\
&= \frac{1}{2\pi i(n+1)} \int_{C'} \frac{1}{\psi(\sigma)^{n+1}} e^{-\sigma} d\sigma \\
&= \frac{1}{n+1} \left(\frac{1}{2\pi i} \int_{C'} \frac{e^{n\sigma} (1 + |\lambda|^2 \sigma)^{n+1}}{\sigma^{n+1}} d\sigma \right) \\
&= \frac{1}{n+1} \text{Res} \left(\frac{e^{n\sigma} (1 + |\lambda|^2 \sigma)^{n+1}}{\sigma^{n+1}}, 0 \right)
\end{aligned}$$

where the second equation is obtained by partial integration and the last equality is obtained by the Residue theorem.

The above Residue is equal to the coefficient of σ^n in the Power series expansion of

$$e^{n\sigma}(1 + |\lambda|^2\sigma)^{-1} = \left(\sum_{k=0}^{\infty} \frac{(n\sigma)^k}{k!} \right) \left(\sum_{i=1}^{n+1} \binom{n+i}{i} (|\lambda|^2\sigma)^i \right).$$

Hence

$$\begin{aligned} \tau((T_\lambda^* T_\lambda)^n) &= \frac{1}{n+1} \sum_{k=0}^n \frac{n^k}{k!} \binom{n+1}{n-k} |\lambda|^{2(n-k)} \\ &= \frac{1}{n+1} \sum_{k=0}^n \frac{n^k}{(k+1)!} \binom{n}{k} |\lambda|^{2n-2k}. \end{aligned}$$

□

4. SPECTRUM AND BROWN-MEASURE OF $T + \sqrt{\epsilon}Y$

Let T be the quasinilpotent DT-operator and let Y be a circular operator $*$ -free from T . In this section we will show, that

$$\sigma(T + \sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \epsilon^{-1})}}\right)$$

and that the Brown-measure $\mu_{T+\sqrt{\epsilon}Y}$ is equal to the uniform distribution on $\overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \frac{1}{\epsilon})}}\right)$, i.e. it has constant density w.r.t. the Lebesgue measure on this disk.

Theorem 4.1. *For every $\epsilon > 0$*

$$(4.1) \quad \sigma(T + \sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \epsilon^{-1})}}\right).$$

Proof. The result can be obtained by the method of Biane and Lehner [BL, Section 5]. Let $a \in \mathbb{C} \setminus \{0\}$. Since $\sigma(T) = \{0\}$ we can write

$$a1 - (T + \sqrt{\epsilon}Y) = \sqrt{\epsilon}\left(\frac{1}{\sqrt{\epsilon}}1 - Y(a1 - T)^{-1}\right)(a1 - T).$$

Hence

$$(4.2) \quad a \notin \sigma(T + \sqrt{\epsilon}Y) \text{ iff } \frac{1}{\sqrt{\epsilon}} \notin \sigma(Y(a1 - T)^{-1}).$$

Let $Y = UH$ be the polar decomposition of Y . Then $Y(a1 - T)^{-1} = UH(a1 - T)^{-1}$, where U is $*$ -free from $H(a1 - T)^{-1}$. Hence $Y(a1 - T)^{-1}$ is R -diagonal. Moreover, since $0 \notin \sigma(Y)$, $Y(a1 - T)^{-1}$ is not invertible, so by [HL, Prop. 4.6.(ii)]

$$(4.3) \quad \sigma(Y(a1 - T)^{-1}) = B(0, \|Y(a1 - T)^{-1}\|_2).$$

By $*$ -freeness of Y and $(a1 - T)^{-1}$ we have

$$(4.4) \quad \|Y(a1 - T)^{-1}\|_2^2 = \|Y\|_2^2 \|(a1 - T)^{-1}\|_2^2 \\ = \|(a1 - T)^{-1}\|_2^2 = \left\| \sum_{n=0}^{\infty} \frac{T^n}{a^{n+1}} \right\|_2^2.$$

Applying now [DH1, lemma 7.2] to $D = 1$ and $\lambda = \frac{1}{a}$ and $\mu = \delta_0$, we get

$$\left\| \sum_{n=0}^{\infty} \frac{T^n}{a^n} \right\|_2^2 = |a|^2 \left(\exp\left(\frac{1}{|a|^2}\right) - 1 \right)$$

Hence by (4.4)

$$\|Y(a1 - T)^{-1}\|_2^2 = \exp\left(\frac{1}{|a|^2}\right) - 1.$$

Thus for $a \in \mathbb{C} \setminus \{0\}$ we get by (4.2) and (4.3)

$$a \notin \sigma(T + \sqrt{\epsilon}Y) \Leftrightarrow \frac{1}{\sqrt{\epsilon}} \notin \sigma(Y(a1 - T)^{-1}) \\ \Leftrightarrow \frac{1}{\sqrt{\epsilon}} > \exp\left(\frac{1}{|a|^2}\right) - 1 \Leftrightarrow |a| > \frac{1}{\sqrt{\log(1 + \frac{1}{\epsilon})}}.$$

Hence $\sigma(T + \sqrt{\epsilon}Y) \cup \{0\} = \overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \frac{1}{\epsilon})}}\right)$. Since $\sigma(T + \sqrt{\epsilon}Y)$ is closed it follows that $\sigma(T + \sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \frac{1}{\epsilon})}}\right)$. \square

In order to compute the Brown measure of $T + \sqrt{\epsilon}Y$, we first observe that $T + \sqrt{\epsilon}Y$ has the same $*$ -distribution as

$$S = \sqrt{a}T_1 + \sqrt{b}T_2^*$$

when T_1 and T_2 are two \mathcal{D} -free quasidiagonal operators and $a = 1 + \epsilon$ and $b = \epsilon$ [Aa1]. We next compute the Brown measure of S for all values of $a, b \in (0, \infty)$.

Lemma 4.2. *Let μ_Q be the Brown measure of an operator Q in a tracial W^* -probability space (M, tr) . Let $r > 0$ and assume that $\mu_Q(\partial B(0, r)) = 0$. Then*

$$\mu_Q(B(0, r)) = -\frac{1}{2\pi} \lim_{\alpha \rightarrow 0^+} \Im \left(\int_{\partial B(0, r)} \text{tr}((Q_\lambda^* Q_\lambda + \alpha 1)^{-1} Q_\lambda^*) d\lambda \right)$$

where $Q_\lambda = Q - \lambda 1$ for $\lambda \in \mathbb{C}$.

Proof. Let $\Delta : M \rightarrow [0, \infty)$ be the Fuglede-Kadison determinant on M , and put $L(\lambda) = \log \Delta(Q_\lambda)$ and

$$L_\alpha(\lambda) = \log \Delta((Q_\lambda^* Q_\lambda + \alpha 1)^{1/2}) = \frac{1}{2} \operatorname{tr}(\log(Q_\lambda^* Q_\lambda + \alpha 1))$$

for $\lambda \in \mathbb{C}$.

Put $\lambda_1 = \Re \lambda$, $\lambda_2 = \Im \lambda$ and let $\nabla^2 = \frac{\partial^2}{\partial \lambda_1^2} + \frac{\partial^2}{\partial \lambda_2^2}$ denote the Laplace operator on \mathbb{C} . Then by [La, Section 2] $\nabla^2 L_\alpha \geq 0$ and for each $\alpha > 0$, the measure

$$(4.5) \quad \mu_\alpha = \frac{1}{2\pi} \nabla^2 L_\alpha(\lambda) d\lambda_1 d\lambda_2$$

is a probability measure on \mathbb{C} . Moreover

$$(4.6) \quad \lim_{\alpha \rightarrow 0} \mu_\alpha = \mu$$

in the weak* topology on $\operatorname{Prob}(\mathbb{C})$. Also from [La, Section 2] the gradient $(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2})$ of L_α is given by

$$(4.7) \quad \frac{\partial}{\partial \lambda_1} L_\alpha(\lambda) = -\Re(\operatorname{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1}))$$

$$(4.8) \quad \frac{\partial}{\partial \lambda_2} L_\alpha(\lambda) = -\Im(\operatorname{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1}))$$

By (4.6)

$$\lim_{\alpha \rightarrow 0} \int_{\mathbb{C}} \phi d\mu_\alpha = \int_{\mathbb{C}} \phi d\mu$$

for all $\phi \in C_0(\mathbb{C})$. Since $1_{B(0,r)}$ is the limit of an increasing sequence $(\phi_n)_{n=1}^\infty$ of $C_0(\mathbb{C})$ -functions with $0 \leq \phi_n \leq 1$ for all $n \in \mathbb{N}$ it follows that

$$\begin{aligned} \mu_Q(B(0,r)) &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}} \phi_n d\mu_Q \\ &= \lim_{n \rightarrow \infty} \left(\lim_{\alpha \rightarrow 0} \int_{\mathbb{C}} \phi_n d\mu_\alpha \right) \leq \lim_{n \rightarrow \infty} \left(\liminf_{\alpha \rightarrow 0} \int_{\mathbb{C}} 1_{B(0,r)} d\mu_\alpha \right) \\ &= \liminf_{\alpha \rightarrow 0} \mu_\alpha(B(0,r)) \end{aligned}$$

Writing $1_{\overline{B}(0,r)}$ as the limit of a decreasing sequence $(\psi_n)_{n=1}^\infty$ of $C_0(\mathbb{C})$ -functions, with $0 \leq \psi_n \leq 1$, one gets in the same way

$$\mu_Q(\overline{B}(0,r)) \geq \limsup_{\alpha \rightarrow 0} \mu_\alpha(\overline{B}(0,r))$$

Hence if $\mu_Q(\partial B(0,r)) = 0$ we have

$$\limsup_{\alpha \rightarrow 0} \mu_\alpha(B(0,r)) \leq \mu_Q(B(0,r)) \leq \liminf_{\alpha \rightarrow 0} \mu_\alpha(B(0,r)),$$

and therefore

$$\mu_Q(B(0, r)) = \lim_{\alpha \rightarrow 0} \mu_\alpha(B(0, r)).$$

Using (4.5) together with Green's theorem applied to the vector-field $(P_\alpha, Q_\alpha) = (-\frac{\partial L_\alpha}{\partial \lambda_2}, \frac{\partial L_\alpha}{\partial \lambda_1})$ we get

$$\begin{aligned} \mu_\alpha(B(0, r)) &= \frac{1}{2\pi} \int_{B(0, r)} \nabla^2 L_\alpha(\lambda) d\lambda_1 d\lambda_2 \\ &= \frac{1}{2\pi} \int_{B(0, r)} \left(\frac{\partial Q_\alpha}{\partial \lambda_1} - \frac{\partial P_\alpha}{\partial \lambda_2} \right) d\lambda_1 d\lambda_2 \\ &= \frac{1}{2\pi} \int_{\partial B(0, r)} P_\alpha d\lambda_1 + Q_\alpha d\lambda_2 \\ &= \frac{1}{2\pi} \int_{\partial B(0, r)} -\frac{\partial L_\alpha}{\partial \lambda_2} d\lambda_1 + \frac{\partial L_\alpha}{\partial \lambda_1} d\lambda_2 \\ &= \Im \left(\frac{1}{2\pi} \int_{\partial B(0, r)} \left(\frac{\partial L_\alpha}{\partial \lambda_1} - i \frac{\partial L_\alpha}{\partial \lambda_2} \right) (d\lambda_1 + i d\lambda_2) \right) \end{aligned}$$

By (4.7) and (4.8)

$$\frac{\partial L_\alpha}{\partial \lambda_1} - i \frac{\partial L_\alpha}{\partial \lambda_2} = -\overline{\text{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1})} = -\text{tr}((Q_\lambda^* Q_\lambda + \alpha 1)^{-1} Q_\lambda^*).$$

Hence

$$\mu_\alpha(B(0, r)) = -\Im \left(\frac{1}{2\pi} \int_{\partial B(0, r)} \text{tr}((Q_\lambda^* Q_\lambda + \alpha 1)^{-1} Q_\lambda^*) d\lambda \right)$$

which completes the proof of the lemma. \square

Let $S = \sqrt{a}T_1 + \sqrt{b}T_2^*$ with $0 < b < a$. Since cS and S have the same $*$ -distribution for all $c \in \mathbb{T}$, the Brown measure μ_S of S is rotation invariant (i.e. invariant under the transformation $z \mapsto cz$, $z \in \mathbb{C}$ when $|c| = 1$). Hence by lemma 4.2 we can compute μ_S , if we can determine

$$\text{tr}((S_\lambda^* S_\lambda + \alpha 1)^{-1} S_\lambda^*)$$

for all $\lambda \in \mathbb{C}$, where $S_\lambda = S - \lambda 1$, and for all α in some interval of the form $(0, \alpha_0)$. This can be done by minor modifications of the methods used in section 3:

Put

$$\tilde{S}_\lambda = \begin{pmatrix} 0 & S_\lambda^* \\ S_\lambda & 0 \end{pmatrix}.$$

Then there exists a $\delta > 0$ (depending on a, b and γ) such that when $\|z\| \leq \delta$ and $|\mu| > \frac{1}{\delta}$ the equality

$$(4.9) \quad \mathcal{R}_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(z) + z^{-1} = \mu 1_2$$

implies that

$$(4.10) \quad z = G_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(\mu 1_2) \\ = (\text{id} \otimes E_{\mathcal{D}}) \begin{pmatrix} \mu(\mu^2 1 - S_\lambda^* S_\lambda)^{-1} & S_\lambda^*(\mu^2 1 - S_\lambda S_\lambda^*)^{-1} \\ S_\lambda(\mu^2 1 - S_\lambda^* S_\lambda)^{-1} & \mu(\mu^2 1 - S_\lambda S_\lambda^*)^{-1} \end{pmatrix}.$$

Moreover, $\tilde{S} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ is $M_2(\mathcal{D})$ -Gaussian by lemma 2.2 since (T_1, T_2^*, T_2, T_2^*) is a \mathcal{D} -Gaussian set. Hence for $z = (z_{ij})_{i,j=1}^2 \in M_2(\mathcal{D})$,

$$\mathcal{R}_{\tilde{S}}^{M_2(\mathcal{D})}(z) = E_{M_2(\mathcal{D})}(\tilde{S} z \tilde{S}) = \begin{pmatrix} E_{\mathcal{D}}(S^* z_{22} S) & 0 \\ 0 & E_{\mathcal{D}}(S z_{11} S^*) \end{pmatrix}.$$

Using that (T_1, T_1^*) and (T_2, T_2^*) have the same \mathcal{D} -distribution as (T, T^*) and that (T_1, T_1^*) and (T_2, T_2^*) are two \mathcal{D} -free sets, we get

$$\begin{aligned} E_{\mathcal{D}}(S^* z_{22} S) &= (aL^* + bL)(z_{22}) \\ E_{\mathcal{D}}(S z_{11} S^*) &= (aL + bL^*)(z_{11}), \end{aligned}$$

where $L(f) : x \mapsto \int_x^1 f(y) dy$ and $L^*(f) : x \mapsto \int_0^x f(y) dy$ for $f \in \mathcal{D}$.

Since $\tilde{S}_\lambda = \tilde{S} - \begin{pmatrix} 0 & \bar{\lambda} 1 \\ \lambda 1 & 0 \end{pmatrix}$ it follows that

$$\mathcal{R}_{\tilde{S}}^{M_2(\mathcal{D})}(z) = \begin{pmatrix} (aL + bL^*)(z_{22}) & -\bar{\lambda} 1 \\ \lambda 1 & (aL^* + bL)(z_{11}) \end{pmatrix}.$$

Thus (4.10) becomes

$$(4.11) \quad \begin{pmatrix} \mu 1 & 0 \\ 0 & \mu 1 \end{pmatrix} \\ = \begin{pmatrix} (aL + bL^*)z_{22} & -\bar{\lambda} 1 \\ \lambda 1 & (aL^* + bL)(z_{11}) \end{pmatrix} + \frac{1}{\det(z)} \begin{pmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{pmatrix}.$$

In analogy with section 3, we look for solutions $z_{ij} \in \mathcal{D} = L^\infty[0, 1]$ of the form

$$(4.12) \quad \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} c_{11} \exp(\sigma x) & c_{12} \\ c_{21} & c_{22} \exp(-\sigma x) \end{pmatrix},$$

where $\sigma \in \mathbb{C}$ and $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \text{GL}(2, \mathbb{C})$. It is easy to check that (4.12) is a solution to (4.11) if the following 5 conditions are fulfilled:

$$\begin{aligned} \det(c) &= \frac{\sigma}{a-b} \\ c_{11} &= \frac{\sigma\mu}{ae^\sigma - b} \\ c_{12} &= -\frac{\sigma\bar{\lambda}}{a-b} \\ c_{21} &= -\frac{\sigma\lambda}{a-b} \\ c_{22} &= \frac{\sigma\mu}{a-be^{-\sigma}} \end{aligned}$$

The first of these conditions is consistent with the remaining 4 if and only if

$$\frac{(\sigma\mu)^2}{(ae^\sigma - b)(a - be^{-\sigma})} - \frac{\sigma^2|\lambda|^2}{(a-b)^2} = \frac{\sigma}{a-b}$$

which is equivalent to

$$(4.13) \quad \mu^2 = \frac{(ae^\sigma - b)(a - be^{-\sigma})(a - b + \sigma|\lambda|^2)}{\sigma(a-b)^2}.$$

Put

$$\sigma_0 := -\min \left\{ \frac{a-b}{|\lambda|^2}, \log\left(\frac{a}{b}\right) \right\}.$$

Then for $\sigma_0 < \sigma < 0$, the right hand side of (4.13) is negative. Let in this case $\mu(\sigma)$ denote the solution to (4.13) with positive imaginary part, i.e.

$$(4.14) \quad \mu(\sigma) = i \frac{ae^{\sigma/2} - be^{-\sigma/2}}{|\sigma|^{1/2}(a-b)} \sqrt{a-b + \sigma|\lambda|^2}$$

for $\sigma_0 < \sigma < 0$. Then with

$$\begin{aligned} c_{11} &= \frac{\sigma\mu(\sigma)}{ae^\sigma - b} & c_{12} &= -\frac{\sigma\bar{\lambda}}{a-b} \\ c_{21} &= -\frac{\sigma\lambda}{a-b} & c_{22} &= \frac{\sigma\mu(\sigma)}{a-be^{-\sigma}} \end{aligned}$$

the matrix $z(\sigma) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ given by (4.12) is a solution to

$$\mathcal{R}_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(z(\sigma)) + z(\sigma)^{-1} = \mu 1_2.$$

By (4.14) $\lim_{\sigma \rightarrow 0^-} |\mu(\sigma)| = \infty$ and $\lim_{\sigma \rightarrow 0^-} |\sigma\mu(\sigma)| = 0$ and therefore $\lim_{\sigma \rightarrow 0^-} \|z(\sigma)\| = 0$.

Hence for some $\sigma_1 \in (\sigma_0, 0)$ we have $|\mu(\sigma)| > \frac{1}{\delta}$ and $\|z(\sigma)\| > \delta$ when $\sigma \in (\sigma_1, 0)$ where $\delta > 0$ is the number described in connection with (4.9). Thus

$$(4.15) \quad z(\sigma) = G_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(\mu(\sigma)1_2)$$

for $\sigma \in (\sigma_1, 0)$. But since both $\sigma \mapsto z(\sigma)$ and $\sigma \mapsto \mu(\sigma)$ are analytic functions (of the real variable σ) it follows that (4.15) holds for all $\sigma \in (\sigma_0, 0)$. Note that $\sigma \mapsto -i\mu(\sigma)$ is a continuous strictly positive function on $(\sigma_0, 0)$, and

$$\lim_{\sigma \rightarrow 0^-} (-i\mu(\sigma)) = +\infty \quad \lim_{\sigma \rightarrow \sigma_0^+} (-i\mu(\sigma)) = 0.$$

Hence for every fixed real number $\alpha > 0$ we can chose $\sigma \in (\sigma_0, 0)$, such that

$$-i\mu(\sigma) = \sqrt{\alpha}.$$

Thus by (4.10) and (4.15)

$$E_{\mathcal{D}}(S_\lambda^*(-\alpha 1 - S_\lambda S_\lambda^*)^{-1}) = z(\sigma)_{12} = -\frac{\sigma \bar{\lambda}}{a - b}$$

which is a constant function in $L^\infty[0, 1]$. Hence

$$\text{tr}(S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1}) = \frac{\sigma \bar{\lambda}}{a - b}$$

from which

$$\int_{\partial B(0, r)} \text{tr}(S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1}) d\lambda = 2\pi i \frac{\sigma r^2}{a - b}$$

when $\sigma_0 < \sigma < 0$, where as before $\sigma_0 = -\min \left\{ \frac{a-b}{|\lambda|^2}, \log\left(\frac{a}{b}\right) \right\}$.

Now $\alpha \rightarrow 0^+$ corresponds to $\sigma \rightarrow \sigma_0^+$. Hence

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} \left(-\frac{1}{2\pi} \Im \int_{\partial B(0, r)} \text{tr}(S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1}) d\lambda \right) \\ = -\frac{\sigma_0 r^2}{a - b} = +\min \left\{ 1, r^2 \frac{\log\left(\frac{a}{b}\right)}{a - b} \right\}. \end{aligned}$$

Observe that $S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1} = (S_\lambda^* S_\lambda + \alpha 1)^{-1} S_\lambda^*$. Thus by lemma 4.2 we have for all but countably many $r > 0$, that

$$\mu_S(B(0, r)) = \min \left\{ 1, r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b} \right\} = \begin{cases} r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b}, & r \leq \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}} \\ 1, & r > \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}} \end{cases}.$$

Since the right hand side is a continuous function of r , the formula actually holds for all $r > 0$. This together with the rotation invariance of μ_S shows, that μ_S is equal to the uniform distribution on $\overline{B}\left(0, \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}}\right)$,

i.e. has constant density $\frac{1}{\pi} \frac{\log\left(\frac{a}{b}\right)}{a-b}$ on this ball, and vanishes outside the ball. Putting $a = 1 + \epsilon$ and $b = \epsilon$ we get in particular

Theorem 4.3. *The Brown measure of $T + \sqrt{\epsilon}Y$ is equal to the uniform distribution on $\overline{B}\left(0, \frac{1}{\sqrt{\log(1+\epsilon^{-1})}}\right)$.*

The Brown measure of $T + \sqrt{\epsilon}Y$ can be used to give an upper bound of the microstate entropy of $T + \sqrt{\epsilon}Y$. By [Šn1] we have for $S \in \mathcal{M}$

$$(4.16) \quad \chi(S) \leq \int_{\mathbb{C}} \int_{\mathbb{C}} \log |z_1 - z_2| d\mu_S(z_1) d\mu_S(z_2) + \frac{5}{4} + \log(\pi \sqrt{2 \text{od}_S})$$

where μ_S is the Brown measure of S on \mathbb{C} and od_S is the off-diagonality of S defined by

$$(4.17) \quad \text{od}_S := \tau(SS^*) - \int_{\mathbb{C}} |z|^2 d\mu_S(z).$$

Lemma 4.4. *For $R > 0$ we have*

$$I := \int_{B(0,R)} \int_{B(0,R)} \log |z_1 - z_2| dz_1 dz_2 = \pi^2 (R^2 \log R - \frac{1}{4})$$

Proof. Polar substitution in I gives

$$I := 4\pi^2 \int_0^R \int_0^R \left(\frac{1}{2\pi} \int_0^{2\pi} \log |r - e^{i\theta}s| d\theta \right) r dr s ds.$$

Let $0 < s < r$. $z \mapsto \log |r - zs|$ is the real value of the complex holomorphic function $z \mapsto \text{Log}(r - zs)$, where Log is the principal branch of the complex logarithm, so $z \mapsto \log |r - zs|$ is a harmonic function in $B(0, \frac{r}{s})$. By the mean value property of harmonic functions

$$\frac{1}{2\pi} \int_0^{2\pi} \log |r - e^{i\theta}s| d\theta = \log(r),$$

so symmetry in r and s reduces I to

$$\begin{aligned} I &:= 4\pi^2 \int_0^R \int_0^R \max\{\log(r), \log(s)\} r dr s ds \\ &= 8\pi^2 \int_0^R \left(\int_0^r \log(r) s ds \right) r dr \\ &= 4\pi^2 \int_0^R r^3 \log(r) dr = \pi^2 R^4 (\log(R) - \tfrac{1}{4}). \end{aligned}$$

□

Theorem 4.5.

$$(4.18) \quad \chi(T + \sqrt{\epsilon}Y) \leq -\frac{1}{2} \log(\log(1 + \epsilon^{-1})) - \frac{1}{4} + \log \pi + \frac{1}{2} \log \left(1 + 2\epsilon - \frac{1}{\log(1 + \epsilon^{-1})} \right).$$

Proof. Let ν_R be the uniform distribution on $\overline{B}(0, R)$. Since ν_R has constant density $(\pi R^2)^{-1}$ on $\overline{B}(0, R)$, we have by lemma 4.4

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \log |z_1 - z_2| d\nu_R(z_1) d\nu_R(z_2) = \log R - \frac{1}{4}.$$

The Brown measure of $S = T + \sqrt{\epsilon}Y$ is $\mu_S = \nu_R$ with $R = \log(1 + \epsilon^{-1})^{-\frac{1}{2}}$, and

$$\text{od}_S = \frac{1}{2} + \epsilon - \int_{\mathbb{C}} |z|^2 d\nu_R = \frac{1}{2} + \epsilon - \frac{R^2}{2}.$$

Hence by (4.16)

$$\chi(T + \sqrt{\epsilon}Y) \leq \log R - \frac{1}{4} + \log \pi + \frac{1}{2} \log(1 + 2\epsilon - R^2).$$

This proves (4.18). □

In [Aa1] the first author proved that the microstate-free analog, $\delta_0^*(T)$, of the free entropy dimension is equal to 2. From Theorem 4.5 one gets only the trivial estimate of the free entropy dimension $\delta_0(T)$, namely

$$(4.19) \quad \delta_0(T) \leq 2 + \lim_{\delta \rightarrow 0^+} \frac{\chi(T + \sqrt{2\delta}Y)}{|\log \delta|} = 2.$$

If $T + \sqrt{\epsilon}Y$ was a DT-operator for all $\epsilon > 0$ then by [Šn1] equality would hold in (4.18), and hence also in (4.19). In the rest of this section, we prove that unfortunately $T + \sqrt{\epsilon}Y$ is not a DT-operator for any $\epsilon > 0$.

If $R = D + T$ is a $\text{DT}(\mu, 1)$ operator it follows from [DH1, lemma 7.2] that for $|\lambda| < \|R\|^{-1}$,

$$\left\| \sum_{n=0}^{\infty} \lambda^n R^n \right\|_2^2 = \frac{1}{|\lambda|^2} \left(\exp \left(\sum_{k,l=1}^{\infty} \lambda^{k+l} \overline{\lambda}^{l+1} M_{\mu}(k, l) \right) - 1 \right),$$

where $M_{\mu}(k, l) = \int_{\sigma(R)} z^k \overline{z}^l d\mu_R(z)$.

If thus μ_D is the uniform distribution on a disk with radius d then

$$M_{\mu_D}(k, l) = 0$$

when $k \neq l$ and

$$\begin{aligned} M_{\mu_D}(k, k) &= \frac{1}{\pi d^2} \int_{B(0,d)} |z|^{2k} dz \\ &= \frac{2\pi}{\pi d^2} \int_0^d r^{2k+1} dr = \frac{2}{d^2} \left[\frac{r^{2k+2}}{2k+2} \right]_0^d = \frac{d^{2k}}{k+1} \end{aligned}$$

for $k \in \mathbb{N}$. Thus

$$\begin{aligned} (4.20) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (D + T)^n \right\|_2^2 &= \frac{1}{|\lambda|^2} \left[\exp \left(\sum_{k=0}^{\infty} |\lambda|^{2(k+1)} \frac{d^{2k}}{k+1} \right) - 1 \right] \\ &= \frac{1}{|\lambda|^2} \exp \left(\frac{1}{d^2} (-\log(1 - d^2 |\lambda|^2)) \right) \\ &= \frac{1}{|\lambda|^2} \left[(1 - d^2 |\lambda|^2)^{-\frac{1}{d^2}} - 1 \right]. \end{aligned}$$

If instead $D + cT$ is a $\text{DT}(\mu_D, c)$ operator with μ_D being the uniform distribution on a disc of radius d then

$$D + cT = c(D' + T)$$

where D' now has the uniform distribution on $B(0, \frac{d}{c})$, so from (4.20) we obtain

$$\begin{aligned} (4.21) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (D + cT)^n \right\|_2^2 &= \left\| \sum_{n=0}^{\infty} (c\lambda)^n (D' + T)^n \right\|_2^2 = \frac{1}{c^2 |\lambda|^2} \left[(1 - d^2 |\lambda|^2)^{-\frac{c^2}{d^2}} - 1 \right]. \end{aligned}$$

Lemma 4.6. *Let $a > b > 0$ and let $S = \sqrt{a}T_1 + \sqrt{b}T_2^*$ where T_1 and T_2 are two \mathcal{D} -free quasidiagonal DT -operators. Then*

$$\left\| \sum_{n=0}^{\infty} \lambda^n S^n \right\|_2^2 = \frac{1}{|\lambda|^2} \frac{e^{(a-b)|\lambda|^2} - 1}{a - be^{(a-b)|\lambda|^2}}, \quad |\lambda| < \frac{1}{\|S\|^2}.$$

Proof. Let $F_n(x) = E_{\mathcal{D}}((S^*)^n S^n)$ for $n \in \mathbb{N}$ and $x \in [0, 1]$. For $t < \frac{1}{\|S\|^2}$ define the \mathcal{D} -valued function

$$(4.22) \quad F(t, x) = \sum_{n=0}^{\infty} F_n(x) t^n.$$

By Speicher's cumulant formula we have by \mathcal{D} -Gaussianity of S that

$$\begin{aligned} F_n &= E_{\mathcal{D}}((S^*)^n S^n) = \sum_{\pi \in \text{NC}(2n)} \kappa_{\pi}^{\mathcal{D}}((S^*)^{\otimes_{\mathcal{B}} n} \otimes_{\mathcal{B}} S^{\otimes_{\mathcal{B}} n}) \\ &= \kappa_2^{\mathcal{D}}(S^* \otimes_{\mathcal{B}} E_{\mathcal{D}}((S^*)^{n-1} S^{n-1}) S) \\ &= (aL^* + bL)(E_{\mathcal{D}}((S^*)^{n-1} S^{n-1})) = (aL^* + bL)(F_{n-1}), \end{aligned}$$

so we get the following recursive algorithm for determining the F_n 's.

$$\begin{cases} F_0(x) = 1 \\ F_n(x) = aL^*(F_{n-1})(x) + bL(F_{n-1})(x), \quad x \in [0, 1] \end{cases},$$

where $L^*(f) : x \mapsto \int_0^x f(y) dy$ and $L(f) : x \mapsto \int_x^1 f(y) dy$. Observe that

$$\frac{d}{dx} L(f)(x) = -f(x) \quad \text{and} \quad \frac{d}{dx} L^*(f)(x) = f(x),$$

and that

$$F_n(0) = aL^*(F_{n-1})(0) + bL(F_{n-1})(0) = b \int_0^1 F_{n-1}(x) dx = b\tau(F_{n-1})$$

for $n \geq 1$. Using (4.22) we have the following differential equation and initial condition in x

$$\begin{cases} \frac{d}{dx} F(t, x) = (a - b)tF(t, x), & x \in [0, 1] \\ F(t, 0) = f(t), \end{cases}$$

where the function f is given by

$$\begin{aligned}
 f(t) &= F(t, 0) = \sum_{n=0}^{\infty} F_n(0)t^n \\
 &= 1 + \sum_{n=1}^{\infty} (aL^*(F_{n-1})(0) + bL(F_{n-1})(0))t^n \\
 &= 1 + b \sum_{n=1}^{\infty} \left(\int_0^1 F_{n-1}(x) dx \right) t^n \\
 &= 1 + bt \int_0^1 \left(\sum_{n=1}^{\infty} F_{n-1}(x)t^{n-1} \right) dx \\
 &= 1 + bt\tau(F(t, \cdot))
 \end{aligned}$$

We thus have the unique solution

$$(4.23) \quad F(t, x) = f(t)e^{(a-b)tx},$$

where we can now use (4.23) and the initial condition to find the function f .

$$\begin{aligned}
 f(t) &= 1 + bt \int_0^1 F(t, x) dx \\
 &= 1 + bt \left[\frac{f(t)}{(a-b)t} e^{(a-b)tx} \right]_0^1 = 1 + bf(t) \frac{(e^{(a-b)t} - 1)}{a-b}.
 \end{aligned}$$

Hence

$$f(t) = \frac{a-b}{a-be^{(a-b)t}}$$

so that

$$F(t, x) = \frac{(a-b)e^{(a-b)tx}}{a-be^{(a-b)t}}.$$

Now observe that

$$\begin{aligned}
 \left\| \sum_{n=0}^{\infty} \lambda^n S^n \right\|_2^2 &= \tau(F(|\lambda|^2, x)) \\
 &= \int_0^1 F(|\lambda|^2, x) dx = \frac{1}{|\lambda|^2} \frac{e^{(a-b)|\lambda|^2} - 1}{a-be^{(a-b)|\lambda|^2}}
 \end{aligned}$$

□

Theorem 4.7. *The operator $T + \sqrt{\epsilon}Y$ is not a DT-operator.*

Proof. By substituting $a = 1 + \epsilon$ and $b = \epsilon$ in lemma 4.6 we have

$$(4.24) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon}Y)^n \right\|_2^2 = \frac{1}{|\lambda|^2} \frac{e^{|\lambda|^2} - 1}{1 + \epsilon - \epsilon e^{|\lambda|^2}}$$

for all λ in a neighborhood of 0. If $T + \sqrt{\epsilon}Y$ is a DT-operator, then by Theorem 4.3 and (4.21), there exists a $c > 0$, such that when $d = \log(1 + \frac{1}{\epsilon})^{-\frac{1}{2}}$

$$(4.25) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon}Y)^n \right\|_2^2 = \frac{1}{c^2 |\lambda|^2} \left((1 - d^2 |\lambda|^2)^{-\frac{c^2}{d^2}} - 1 \right)$$

for all λ in a neighborhood of 0. Consider the two analytic functions,

$$\begin{aligned} f(s) &= \frac{e^s - 1}{1 + \epsilon - \epsilon e^s}, \\ g(s) &= \frac{1}{c^2} \left((1 - d^2 s)^{-\frac{c^2}{d^2}} - 1 \right) \end{aligned}$$

which are both defined in the complex disc $U = B(0, \log(1 + \frac{1}{\epsilon})^{-\frac{1}{2}})$. By (4.24) and (4.25) $f(s) = g(s)$ for s in some real interval of the form $(0, \delta)$ and hence $f(s) = g(s)$ for all $s \in U$. Moreover f has a meromorphic extension to the full complex plane with a simple pole at $s_0 = \log(1 + \frac{1}{\epsilon})$. Hence g also has a meromorphic extension to the full complex plane with a simple pole at $\log(1 + \frac{1}{\epsilon}) = d^{-2}$. This implies $c = d$. In this case

$$g(s) = \frac{1}{d^2} ((1 - d^2 s)^{-1} - 1)$$

which is analytic in $\mathbb{C} \setminus \{s_0\}$. However f has infinitely many poles, namely

$$s_p = \log \left(1 + \frac{1}{\epsilon} \right) + p2\pi, \quad p \in \mathbb{Z}.$$

Since the meromorphic extensions of f and g must coincide, we have reached a contradiction. Therefore $T + \sqrt{\epsilon}Y$ is not a DT-operator. \square

5. ŚNIADY'S MOMENT FORMULAS. THE CASE $k = 2$.

Let $k \in \mathbb{N}$ be fixed, and let $(P_{k,n})_{n=0}^{\infty}$ be the sequence of polynomials defined recursively by

$$(5.1) \quad \begin{cases} P_{k,n}(x) = 1, \\ P_{k,n}^{(k)}(x) = P_{k,n-1}(x+1), & n = 1, 2, \dots, \\ P_{k,n}(0) = P_{k,n}^{(1)}(0) = \dots P_{k,n}^{(k-1)}(0) = 0, & n = 1, 2, \dots \end{cases}$$

where $P_{k,n}^{(l)}$ denotes the l 'th derivative of $P_{k,n}$. As in the previous sections, T denotes the quasinilpotent DT operator. Śniady's main results from [Śn2] are:

Theorem 5.1. [Śn2, Theorem 5 and Theorem 7]

(a) For all $k, n \in \mathbb{N}$:

$$(5.2) \quad E_{\mathcal{D}} \left(((T^*)^k T^k)^n \right) (x) = P_{k,n}(x), \quad x \in [0, 1].$$

(b) For all $k, n \in \mathbb{N}$:

$$(5.3) \quad \tau \left(((T^*)^k T^k)^n \right) = \frac{n^{nk}}{(nk+1)!}$$

Actually Śniady considers $E_{\mathcal{D}}((T^k(T^*)^k)^n)$ instead of $E_{\mathcal{D}}(((T^*)^k T^k)^n)$, but it is easily seen, that Theorem 5.1 (a) is equivalent to [Śn2, Theorem 5], by the simple change of variable $x \mapsto 1 - x$.

Śniady's proof of Theorem 5.1 is a very technical combinatorial proof. In this and the following section we will give an analytical proof of Theorem 5.1 based on Voiculescu's \mathcal{R} -transform with amalgamation.

As in [DH2, (2.11)] we put

$$\rho(z) = -W_0(-z), \quad z \in \mathbb{C} \setminus [\tfrac{1}{e}, \infty),$$

where W_0 is the principal branch of Lambert's W-function. Then ρ is the principal branch of the inverse function of $z \mapsto ze^{-z}$. We shall need the following result from [DH2, Prop. 4.2].

Lemma 5.2. [DH2, Prop. 4.2] *Let $(P_{k,n})_{n=0}^{\infty}$ be a sequence of polynomials given by (5.1). Put for $s \in \mathbb{C}$, $|s| < \frac{1}{e}$ and $j = 1, \dots, k$*

$$(5.4) \quad \alpha_j(s) = \rho \left(s e^{i \frac{2\pi j}{k}} \right),$$

$$(5.5) \quad \gamma_j(s) = \begin{cases} \prod_{l \neq j} \frac{\alpha_l(s)}{\alpha_l(s) - \alpha_j(s)}, & 0 < |s| < \frac{1}{e} \\ \frac{1}{k}, & s = 0. \end{cases}$$

Then

$$(5.6) \quad \sum_{n=0}^{\infty} (ks)^{nk} P_{k,n}(x) = \sum_{j=1}^k \gamma_j(s) e^{k\alpha_j(s)x}$$

for all $x \in \mathbb{R}$ and all $s \in B(0, \frac{1}{e})$.

The case $k = 1$ of theorem 5.1 is the special case $\lambda = 0$ of theorem 3.2. To illustrate our method of proof of theorem 5.1 for $k \geq 2$, we first consider the case $k = 2$.

Define $\tilde{T} \in M_4(\mathcal{A})$ by

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & T^* \\ T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & T^* & 0 \end{pmatrix}.$$

Then $\|\tilde{T}\| = \|T\| = \sqrt{e}$. (cf. [DH1, Corollary 8.11]) For $\mu \in \mathbb{C}$, $|\mu| < \frac{1}{e}$ we let $z = z(\mu)$, denote the Cauchy transform of \tilde{T} at $\tilde{\mu} = \mu 1_{M_4(\mathcal{A})}$ wrt. amalgamation over $M_4(\mathcal{D})$ i.e.

$$z = E_{\mathcal{D}} \left((\tilde{\mu} - \tilde{T})^{-1} \right).$$

Clearly

$$(5.7) \quad (\tilde{\mu} - \tilde{T})^{-1} = \sum_{n=0}^{\infty} \mu^{-n-1} \tilde{T}^n = \left(\sum_{n=0}^3 \mu^{-n-1} \tilde{T}^n \right) \left(\sum_{n=0}^{\infty} \mu^{-4n} \tilde{T}^{4n} \right).$$

By direct computation

$$\tilde{T}^2 = \begin{pmatrix} 0 & 0 & (T^*)^2 & 0 \\ 0 & 0 & 0 & TT^* \\ T^2 & 0 & 0 & 0 \\ 0 & T^*T & 0 & 0 \end{pmatrix},$$

$$\tilde{T}^3 = \begin{pmatrix} 0 & (T^*)^2T & 0 & 0 \\ 0 & 0 & T(T^*)^2 & 0 \\ 0 & 0 & 0 & T^2T^* \\ T^*T^2 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{T}^4 = \begin{pmatrix} (T^*)^2T^2 & 0 & 0 & 0 \\ 0 & T(T^*)^2T & 0 & 0 \\ 0 & 0 & T^2(T^*)^2 & 0 \\ 0 & 0 & 0 & T^*T^2T^* \end{pmatrix}.$$

Hence using the fact that the expectation $E_{\mathcal{D}}$ of a monomial in T and T^* vanishes unless T and T^* occur the same number of times, we get from (5.7) that z is of the form

$$(5.8) \quad z = \begin{pmatrix} z_{11} & 0 & 0 & 0 \\ 0 & z_{22} & 0 & z_{24} \\ 0 & 0 & z_{33} & 0 \\ 0 & z_{42} & 0 & z_{44} \end{pmatrix}$$

where $z_{11}, z_{22}, z_{24}, z_{33}, z_{42}, z_{44} \in \mathcal{D}$ are given by

$$\begin{aligned} z_{11} &= \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-4}(T^*)^2 T^2)^{-1}), \\ z_{22} &= \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-4} T (T^*)^2 T)^{-1}), \\ z_{33} &= \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-4} T^2 (T^*)^2)^{-1}), \\ z_{44} &= \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-4} T^* T^2 T^*)^{-1}), \\ z_{24} &= \mu^{-3} E_{\mathcal{D}}(T(1 - \mu^{-4}(T^*)^2 T^2)^{-1} T^*), \\ z_{42} &= \mu^{-3} E_{\mathcal{D}}(T^*(1 - \mu^{-4} T^2 (T^*)^2)^{-1} T). \end{aligned}$$

For the last 2 identities, we have used, that

$$A(1 - \eta BA)^{-1} = (1 - \eta AB)^{-1} A$$

for $A, B \in \mathcal{A}$ and $\eta \in \mathbb{C}$ whenever both sides of this equality are welldefined.

By lemma 2.1, we know, that there exists a $\delta > 0$ such that when $w \in M_4(\mathcal{D})_{\text{inv}}$ and $\mu \in \mathbb{C}$ satisfies $\|w\| < \delta$, $|\mu| > \frac{1}{\delta}$ and

$$(5.9) \quad \mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_4(\mathcal{A})}$$

then $w = E_{M_4(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}) = z$. In particular

$$w_{11} = z_{11} = \mu^{-1}((1 - \mu^{-4}(T^*)^2 T^2)^{-1}),$$

Hence, if we can find a suitable solution to (5.8) for all $\mu \in \mathbb{C}$ in a neighborhood of ∞ , we can find $E_{\mathcal{D}}(((T^*)^2 T^2)^n)$ for $n = 1, 2, \dots$ by determining the power series expansion of w_{11} as a function of μ^{-1} .

Since (T, T^*) is a \mathcal{D} -Gaussian pair by [DH2, Appendix] it follows from lemma 2.2 that

$$\kappa_n^{M_4(\mathcal{D})}((m_1 \otimes a_1) \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} (m_n \otimes a_n)) = 0$$

when $n \neq 2$, $m_1, m_2, \dots, m_n \in M_4(\mathbb{C})$ and $a_1, a_2, \dots, a_n \in \{T, T^*\}$. By definition

$$\tilde{T} = (e_{21} + e_{32}) \otimes T + (e_{43} + e_{14}) \otimes T^*$$

so by linearity of $\kappa_n^{M_4(\mathcal{D})}$, it follows that

$$\kappa_n^{M_4(\mathcal{D})}(\tilde{T} \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} \tilde{T}) = 0$$

when $n \neq 2$ i.e. \tilde{T} is $M_4(\mathcal{D})$ -Gaussian.

Hence using (2.4) we get

$$\begin{aligned}\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) &= \kappa_2^{M_4(\mathcal{D})}(\tilde{T} \otimes_{M_4(\mathcal{D})} w \tilde{T}) = E_{M_4(\mathcal{D})}(\tilde{T} w \tilde{T}) \\ &= E_{M_4(\mathcal{D})} \left(\begin{pmatrix} T^* w_{42} T & 0 & T^* w_{44} T^* & 0 \\ 0 & 0 & 0 & T w_{11} T^* \\ T w_{22} T & 0 & T w_{24} T^* & 0 \\ 0 & T^* w_{33} T & 0 & 0 \end{pmatrix} \right)\end{aligned}$$

for $w = (w_{ij})_{i,j=1,\dots,4} \in M_4(\mathcal{D})$.

Since $E_{\mathcal{D}}(TfT) = E_{\mathcal{D}}(T^*fT^*) = 0$, and $E_{\mathcal{D}}(T^*fT) = L^*(f)$, $E_{\mathcal{D}}(TfT^*) = L(f)$ for $f \in L^\infty([0, 1])$, we have:

$$\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) = \begin{pmatrix} L^*(w_{42}) & 0 & 0 & 0 \\ 0 & 0 & 0 & L(w_{11}) \\ 0 & 0 & L(w_{24}) & 0 \\ 0 & L^*(w_{33}) & 0 & 0 \end{pmatrix}$$

for $w \in M_4(\mathcal{D})$. By (5.8) we only have to consider w of the form

$$(5.10) \quad w = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ 0 & w_{22} & 0 & w_{24} \\ 0 & 0 & w_{33} & 0 \\ 0 & w_{42} & 0 & w_{44} \end{pmatrix}.$$

For $w \in M_4(\mathcal{D})_{\text{inv}}$ of the form (5.10), (5.9) reduces to the three equations

$$(5.11) \quad \begin{cases} L^*(w_{42}) + \frac{1}{w_{11}} = \mu 1_{\mathcal{D}} \\ \begin{pmatrix} 0 & L(w_{11}) \\ L^*(w_{33}) & 0 \end{pmatrix} + \begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathcal{D})} \\ L(w_{24}) + \frac{1}{w_{33}} = \mu 1_{\mathcal{D}} \end{cases}.$$

Definition 5.3. Let $f \in C([0, 1])$. We call $(f^{(-n)})_{n=1}^l$ for the successive antiderivatives of f if

$$\frac{d}{dx}(f^{(-n)}) = f^{(1-n)} \text{ for } n = 2, 3, \dots, l$$

and

$$\frac{d}{dx}(f^{(-1)}) = f.$$

Lemma 5.4. Let $f \in C^2([0, 1])$ and let $f^{(-1)}$ and $f^{(-2)}$ be the successive antiderivatives of f for which

$$(i) \quad f^{(-1)}(1) = 0, \quad f^{(-2)}(1) = \mu^3.$$

Assume further, that

(ii) $f(0) = \mu^{-1}$ and $f^{(1)}(0) = 0$.

(iii) For all $x \in [0, 1]$,

$$\begin{aligned} f(x) &\neq 0 \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} &\neq 0 \end{aligned}$$

while

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{vmatrix} = 0$$

Then $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42} \in C([0, 1])$ given by

$$(5.12) \quad \left\{ \begin{aligned} w_{11} &= f \\ w_{22} = w_{44} &= -\frac{1}{\mu} \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \\ w_{24} &= \frac{1}{\mu^2} \frac{f^{(-1)} \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \\ w_{42} &= \frac{f^{(1)}}{f^2} \\ w_{33} &= \mu^2 \frac{f \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} \end{aligned} \right.$$

is a solution to (5.11). Moreover

$$(5.13) \quad \begin{vmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{vmatrix} = -\frac{1}{\mu^2} \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2}$$

and

$$(5.14) \quad \left\{ \begin{array}{l} L(w_{11}) = -f^{(-1)} \\ L(w_{24}) = \mu - \frac{1}{\mu^2} \frac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f} \\ L^*(w_{42}) = \mu - \frac{1}{f} \\ L^*(w_{33}) = -\mu^2 \frac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}} \end{array} \right. .$$

Proof. Assume $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}$ is given by (5.12). Then (5.13) follows immediately. Note that for $f \in C([0, 1])$, the functions $g = L(f)$ and $h = L^*(f)$ are characterized by

$$\begin{aligned} g^{(1)} &= -f & \text{and} & & g(1) &= 0 \\ h^{(1)} &= f & \text{and} & & h(0) &= 0. \end{aligned}$$

Hence (5.14) is equivalent to (5.15) and (5.16) below.

$$(5.15) \quad \left\{ \begin{array}{l} \frac{d}{dx} f^{(-1)} = w_{11} \\ \frac{d}{dx} \left(\frac{1}{\mu^2} \frac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f} \right) = w_{24} \\ \frac{d}{dx} \left(-\frac{1}{f} \right) = w_{42} \\ \frac{d}{dx} \left(-\mu^2 \frac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}} \right) = w_{33} \end{array} \right.$$

$$(5.16) \quad \left\{ \begin{array}{l} f^{(-1)}(1) = 0, \quad \frac{\begin{vmatrix} f^{(-2)}(1) & f^{(-1)}(1) \\ f^{(-1)}(1) & f(1) \end{vmatrix}}{f(1)} = \mu^3 \\ \frac{1}{f(0)} = \mu, \quad f^{(1)}(0) = 0 \end{array} \right. .$$

Now, (5.16) is trivial from (i) and (ii). Next we prove (5.15): Clearly

$$\frac{d}{dx} f^{(-1)} = f = w_{11} \quad \text{and} \quad \frac{d}{dx} \left(-\frac{1}{f}\right) = \frac{f^{(1)}}{f^2} = w_{42}.$$

Moreover

$$\begin{aligned} (5.17) \quad & \frac{d}{dx} \left(\frac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f} \right) \\ &= \frac{f \begin{vmatrix} f^{(-2)} & f \\ f^{(-1)} & f^{(1)} \end{vmatrix} - f^{(1)} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f^2} = \frac{f^{(-1)} \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} = \mu^2 w_{24} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx} \left(\frac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}} \right) &= \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix} f^{(2)} - \begin{vmatrix} f^{(-1)} & f \\ f^{(1)} & f^{(2)} \end{vmatrix} f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} \\ &= -\frac{f \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} = -\frac{1}{\mu^2} w_{33}. \end{aligned}$$

Hence (5.15) holds. It remains to be proved that $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}$ is a solution to (5.11). By (5.12) and (5.14), we have

$$L^*(w_{42}) + \frac{1}{w_{11}} = \left(\mu - \frac{1}{f} \right) + \frac{1}{f} = \mu.$$

Moreover by (5.12) and (5.13)

$$\begin{aligned} \begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} &= \frac{1}{w_{22}w_{44} - w_{24}w_{42}} \begin{pmatrix} w_{44} & -w_{24} \\ -w_{42} & w_{22} \end{pmatrix} \\ &= \begin{pmatrix} \mu & f^{(-1)} \\ \mu^2 \frac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}} & \mu \end{pmatrix} \end{aligned}$$

which proves that the first and the second inequality in (5.11).

By (5.12) and (5.14),

$$w_{33}(\mu - L(w_{24})) = \frac{\begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} = 1 + \frac{\sigma}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2}$$

where

$$\sigma = \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix} - \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2 = f \begin{vmatrix} f^{(-2)} & f^{(-1)} & f \\ f^{(-1)} & f & f^{(1)} \\ f & f^{(1)} & f^{(2)} \end{vmatrix}.$$

Hence by (iii), $\sigma = 0$. Therefore $w_{33}(x) \neq 0$ for all $x \in [0, 1]$ and $w_{33}^{-1} = \mu - L(w_{24})$, proving the last equality in (5.11). \square

Lemma 5.5. *Let $\alpha_j(s), \gamma_j(s)$ for $j = 1, 2$ be as in lemma 5.2 for $k = 2$, i.e. $\alpha_1(0) = \alpha_2(0) = 0$, $\gamma_1(0) = \gamma_2(0) = \frac{1}{2}$ and for $0 < |s| < e^{-1}$:*

$$\begin{aligned} \alpha_1(s) &= \rho(s), & \alpha_2(s) &= \rho(-s), \\ \gamma_1(s) &= \frac{\alpha_1(s)}{\alpha_1(s) - \alpha_2(s)}, & \gamma_2(s) &= \frac{\alpha_2(s)}{\alpha_2(s) - \alpha_1(s)}. \end{aligned}$$

Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{e}$, put $s = \frac{1}{2}\mu^{-2}$ and

$$(5.18) \quad f(x) = \frac{1}{\mu} \left(\sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x} \right), \quad x \in \mathbb{R}$$

$$(5.19) \quad f^{(-1)}(x) = \frac{1}{2\mu} \left(\sum_{j=1}^2 \frac{\gamma_j(s)}{\alpha_j(s)} e^{2\alpha_j(s)x} \right), \quad x \in \mathbb{R}$$

$$(5.20) \quad f^{(-2)}(x) = \frac{1}{4\mu} \left(\sum_{j=1}^2 \frac{\gamma_j(s)}{\alpha_j(s)^2} e^{2\alpha_j(s)x} \right), \quad x \in \mathbb{R}$$

Then

(i) $f^{(-1)}, f^{(-2)}$ are succesively antiderivatives of f ,

$$(5.21) \quad f^{(-1)}(1) = 0, \quad f^{(-2)}(1) = \mu^3$$

and

$$(5.22) \quad f(0) = \mu^{-1}, \quad f^{(1)}(0) = 0.$$

(ii) The following asymptotic formulas holds for $|\mu| \rightarrow \infty$:

$$\begin{aligned} f^{(-2)}(x) &= \mu^3 + \mathcal{O}(\mu^{-1}) \\ f^{(-1)}(x) &= (x-1)\mu^{-1} + \mathcal{O}(\mu^{-5}) \\ f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5}) \\ f^{(1)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}) \\ f^{(2)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}) \end{aligned}$$

where the error estimates holds uniformly in x on a compact subset in \mathbb{R} .

(iii) There exists $\mu_0 \geq \sqrt{e}$ such that the restriction of f to $[0, 1]$ satisfies all the conditions in lemma 5.4, when $|\mu| > \mu_0$.

Proof. Clearly $f^{(-1)}$ and $f^{(-2)}$ are succesively antiderivatives of f and

$$\begin{aligned} f(0) &= \frac{1}{\mu} \sum_{j=1}^2 \gamma_j(s) = \frac{1}{\mu} \\ f^{(1)}(0) &= \frac{2}{\mu} \sum_{j=1}^2 \alpha_j(s) \gamma_j(s) = 0. \end{aligned}$$

To prove (5.21), note first, that since $\rho : \mathbb{C} \setminus [\frac{1}{e}, \infty) \rightarrow \mathbb{C}$ is a branch of the inverse function of $z \mapsto ze^{-z}$, we have

$$\rho(w)e^{-\rho(w)} = w, \quad |w| < \frac{1}{e}$$

and therefore

$$e^{2\alpha_j(s)} = \frac{\alpha_j(s)^2}{s^2}, \quad j = 1, 2.$$

Since $s^2 = \frac{1}{4}\mu^{-4}$, it follows that

$$(5.23) \quad f^{(-2)}(x+1) = \mu^4 f(x), \quad x \in \mathbb{R}$$

$$(5.24) \quad f^{(-1)}(x+1) = \mu^4 f^{(1)}(x), \quad x \in \mathbb{R}$$

$$(5.25) \quad f(x+1) = \mu^4 f^{(2)}(x), \quad x \in \mathbb{R}.$$

In particular

$$\begin{aligned} f^{(-2)}(1) &= \mu^4 f(0) = \mu^3 \\ f^{(-1)}(1) &= \mu^4 f^{(1)}(0) = 0. \end{aligned}$$

By the proof of [DH2, Prop. 4.2], $\alpha_j(s)$ and $\rho_j(s)$ are continuous functions of $s \in B(0, \frac{1}{e})$. Hence, regarding f as a function of μ ,

$$\lim_{|\mu| \rightarrow \infty} (\mu f(x)) = \sum_{j=1}^2 \gamma_j(0) e^{2\alpha_j(0)x} = 1$$

where the limit holds uniformly in x on compact subsets of \mathbb{R} . Hence by (5.25) $f^{(2)}(x) = \mathcal{O}(\mu^{-5})$ as $|\mu| \rightarrow \infty$ uniformly in x on compact subsets of \mathbb{R} . By (5.22),

$$(5.26) \quad f^{(1)}(x) = \int_0^x f^{(2)}(t) dt$$

$$(5.27) \quad f(x) = \mu^{-1} + \int_0^x f^{(1)}(t) dt$$

which implies, that $f^{(1)}(x) = \mathcal{O}(\mu^{-5})$ and

$$(5.28) \quad f(x) = \mu^{-1} + \mathcal{O}(\mu^{-5})$$

uniformly in x on compact subsets of \mathbb{R} .

Using again (5.25), (5.26) and (5.27), we get

$$\begin{aligned} f^{(2)}(x) &= \mu^{-5} + \mathcal{O}(\mu^{-9}) \\ f^{(1)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}). \end{aligned}$$

By (5.21)

$$\begin{aligned} f^{(-1)}(x) &= \int_1^x f(t) dt \\ f^{(-2)}(x) &= \mu^3 + \int_1^x f^{(-1)}(t) dt. \end{aligned}$$

Hence by (5.28),

$$\begin{aligned} f^{(-1)}(x) &= (x-1)\mu^{-1} + \mathcal{O}(\mu^{-5}) \\ f^{(-2)}(x) &= \mu^3 + \mathcal{O}(\mu^{-1}) \end{aligned}$$

where all estimates holds uniformly on compact subsets of \mathbb{R} . This proves (ii).

By (i), $f^{(-1)}, f^{(-2)}$ coincide with the successive antiderivatives of f considered in lemma 5.4 and $f(0) = \mu^{-1}, f^{(1)}(0) = 0$.

Moreover, by (ii),

$$\begin{aligned} f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5}) \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} &= \mu^{-2} + \mathcal{O}(\mu^{-6}) \end{aligned}$$

where the error terms holds uniformly in $x \in [0, 1]$. Hence there exists $\mu_0 \geq \sqrt{e}$, such that

$$f(x) \neq 0 \text{ and } \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} \neq 0$$

for all $x \in [0, 1]$. Moreover by the matrix factorization

$$(5.29) \quad \begin{pmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{pmatrix} \\ = \begin{pmatrix} 1 & 1 \\ 2\alpha_1(s) & 2\alpha_2(s) \\ 4\alpha_1(s)^2 & 4\alpha_2(s)^2 \end{pmatrix} \begin{pmatrix} \frac{\gamma_1(s)}{4\alpha_1(s)^2} e^{2\alpha_1(s)x} & 0 \\ 0 & \frac{\gamma_2(s)}{4\alpha_2(s)^2} e^{2\alpha_2(s)x} \end{pmatrix} \begin{pmatrix} 1 & 2\alpha_1(s) & 4\alpha_1(s)^2 \\ 1 & 2\alpha_2(s) & 4\alpha_2(s)^2 \end{pmatrix}$$

it follows, that the matrix on the left hand side has rank less than or equal to 2, i.e.

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{vmatrix} = 0$$

for $x \in [0, 1]$. Hence f satisfies all the conditions in lemma 5.4, when $|\mu| > \mu_0$. \square

Proof of Theorem 5.1 in the case $k = 2$: By lemma 2.1 there exists a $\delta > 0$, such that when $w \in M_4(\mathcal{D})_{\text{inv}}$ and $\mu \in \mathbb{C}$ satisfies $\|w\| < \delta$, $|\mu| > \frac{1}{\delta}$ and

$$(5.30) \quad \mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_4(\mathcal{D})}$$

then $w = E_{\mathcal{D}}((\tilde{\mu} - \tilde{T})^{-1})$. In particular

$$(5.31) \quad w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-4}(T^*)^2 T^2)^{-1}).$$

Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{e}$, put $s = \frac{1}{2}\mu^{-2}$ and

$$f(x) = \frac{1}{\mu} \left(\sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x} \right)$$

for $x \in [0, 1]$ as in lemma 5.5. By lemma 5.5 (iii) there exists a $\mu_0 > \sqrt{e}$, such that when $|\mu| > \mu_0$, then f satisfies all the requirements of lemma 5.4. Hence by lemma 5.4, the matrix $w \in M_4(\mathcal{D})$ given by (5.10) and (5.12) is a solution to (5.30). Moreover by the asymptotic formulas in lemma 5.5 (ii),

$$\begin{aligned} \begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) \\ f^{(-1)}(x) & f(x) \end{vmatrix} &= \mu^2 + \mathcal{O}(\mu^{-2}), \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f'(x) \end{vmatrix} &= -\mu^{-2} + \mathcal{O}(\mu^{-6}), \\ \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} &= \mu^{-6} + \mathcal{O}(\mu^{-10}). \end{aligned}$$

Hence by (5.12) and the asymptotic formulas for $f^{(-1)}$, f and f' , we have

$$\begin{aligned} w_{11} &= \mu^{-1} + \mathcal{O}(\mu^{-5}), \\ w_{22} &= w_{44} = \mu^{-1} + \mathcal{O}(\mu^{-5}), \\ w_{24} &= (1-x)\mu^{-3} + \mathcal{O}(\mu^{-3}), \\ w_{42} &= x\mu^{-3} + \mathcal{O}(\mu^{-3}), \\ w_{33} &= \mu^{-1} + \mathcal{O}(\mu^{-5}), \end{aligned}$$

where all the error estimates holds uniformly in $x \in [0, 1]$. Hence, there exists $\mu_1 \geq \max\{\mu_0, \frac{1}{\delta}\}$, such that when $|\mu| > \mu_1$ then $\|w\| < \delta$, and hence

$$w = E_{M_4(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

By (5.12), $w_{11} = f$. Hence by (5.31) and (5.18)

$$E_{\mathcal{D}}((1 - \mu^{-4}(T^*)^2 T^2)^{-1})(x) = \mu f(x) = \sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x}$$

where $s = \frac{1}{2}\mu^{-2}$, i.e. for $|s| < \frac{1}{2}\mu_1^{-2}$,

$$E_{\mathcal{D}}((1 - (2s)^2(T^*)^2 T^2)^{-1})(x) = \sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x}$$

and therefore

$$(5.32) \quad \sum_{j=0}^{\infty} (2s)^{2n} E_{\mathcal{D}}(((T^*)^2 T^2)^n)(x) = \sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x}.$$

Hence by lemma 5.2 and by the uniqueness of the power series expansions of analytic functions, we have

$$E_{\mathcal{D}}(((T^*)^2 T^2)^n)(x) = P_{2,n}(x)$$

for $n \in \mathbb{N}$ and $x \in [0, 1]$. This proves theorem 5.1(a) in the case $k = 2$. Theorem 5.1 (b) also follows from (5.32) by integrating the right hand side of (5.32) from 0 to 1 with respect to x (cf. [DH2, remark 4.3]). \square

6. ŠNIADY'S MOMENT FORMULAS. THE GENERAL CASE.

The above proof of Theorem 5.1 in the case $k = 2$ can fairly easily be generalized to all $k \geq 2$ (Recall that the case $k = 1$ is contained in theorem 3.2).

Let $k \geq 2$ and define $\tilde{T} \in M_{2k}(\mathcal{A})$ by

$$\tilde{T} = \sum_{j=1}^k (T \otimes e_{j+1,j} + T^* \otimes e_{k+j+1,k+j})$$

where the indices are computed modulo $2k$, such that $e_{2k+1,2k} = e_{1,2k}$. For $\mu \in \mathbb{C}$, $|\mu| < \frac{1}{\sqrt{e}}$, we put $\tilde{\mu} = \mu 1_{2k}$ and

$$z = z(\mu) = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

Then only the diagonal entries $z_{11}, \dots, z_{2k,2k}$ and the off-diagonal entries $z_{2,2k}, z_{3,2k-1}, \dots, z_{2k,2}$ can be non-zero. Moreover,

$$z_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k} (T^*)^k T^k)^{-1}).$$

The operator \tilde{T} is $M_{2k}(\mathcal{D})$ -Gaussian, and repeating the arguments for $k = 2$, we get that for $w \in M_{2k}(\mathcal{D})$, the matrix

$$(6.1) \quad u = \mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w)$$

can have at most $2k$ non-zero entries, namely the entries

$$(6.2) \quad \begin{aligned} u_{11} &= L^*(w_{2k,2}) \\ u_{2k,2} &= L^*(w_{2k-1,3}) \\ &\vdots \\ u_{k+2,k} &= L^*(w_{k+1,k+1}) \\ u_{k+1,k+1} &= L(w_{k,k+2}) \\ u_{k,k+2} &= L(w_{k-1,k+3}) \\ &\vdots \\ u_{2,2k} &= L(w_{1,1}). \end{aligned}$$

By lemma 2.1 there exists a $\delta > 0$ (depending on k), such that if $w \in M_{2k}(\mathcal{D})_{\text{inv}}$, $\|w\| < \delta$, $\mu \in \mathbb{C}$, $|\mu| > \frac{1}{\delta}$ and

$$(6.3) \quad \mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_{2k}(\mathcal{D})},$$

then

$$w = z = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

In particular

$$w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k} (T^*)^k T^k)^{-1}).$$

Next we construct an explicit solution to (6.3). By the above remarks on z , it is sufficient to consider those $w \in M_{2k}(\mathcal{D})_{\text{inv}}$ for which only the

entries $z_{11}, \dots, z_{2k,2k}$ and $z_{2,2k}, z_{3,2k-1}, \dots, z_{2k,2}$ can be non-zero. For such w , (6.3) can by (6.1) and (6.2) be reduced to the $k+1$ identities:

$$(6.4) \quad \begin{cases} L^*(w_{2k,2}) + \frac{1}{w_{11}} = \mu 1_{\mathcal{D}} \\ \left(\begin{pmatrix} 0 & L(w_{j+1,2k+1-j}) \\ L^*(w_{2k-1-j,j+3}) & 0 \end{pmatrix} + \begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} \right) = \mu 1_{M_2(\mathcal{D})}, \\ L(w_{k,k+2}) + \frac{1}{w_{k+1,k+1}} = \mu 1_{\mathcal{D}}. \end{cases} \quad j = 0, 1, \dots, k-2,$$

Definition 6.1. For $j \in \mathbb{N} \cup \{0\}$ and $g \in C^{2j+2}$, we let $\Delta_j(g)$ denote the determinant

$$(6.5) \quad \Delta_j(g) = \begin{vmatrix} g & g^{(1)} & \cdots & g^{(j)} \\ g^{(1)} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & g^{(2j-1)} \\ g^{(j)} & \ddots & g^{(2j-1)} & g^{(2j)} \end{vmatrix}.$$

In particular $\Delta_0(g) = g$.

Lemma 6.2. Let $g \in C^{2j+2}(\mathbb{R})$ and $j \in \mathbb{N}$. Then

$$(6.6) \quad \Delta_j(g^{(2)})\Delta_j(g) - \Delta_j(g^{(1)})^2 = \Delta_{j-1}(g^{(2)})\Delta_{j+1}(g)$$

and

$$(6.7) \quad \Delta_{j-1}(g^{(2)}) \frac{d}{dx} (\Delta_j(g)) - \Delta_j(g) \frac{d}{dx} (\Delta_{j-1}(g^{(2)})) = \Delta_{j-1}(g^{(1)})\Delta_j(g^{(1)}).$$

The proof of lemma 6.2 relies on elementary matrix manipulations and is contained in lemma A.1 of appendix A. More specifically (6.6) is a direct consequence of (a) from lemma A.1, and (6.7) follows from (b) of lemma A.1 by using the elementary fact that:

$$\frac{d}{dx} (\Delta_j(g)) = \begin{vmatrix} g & g^{(1)} & \cdots & g^{(j)} \\ g^{(1)} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & g^{(2j-1)} \\ g^{(j-1)} & \ddots & g^{(2j-2)} & g^{(2j-1)} \\ g^{(j+1)} & \ddots & g^{(2j)} & g^{(2j+1)} \end{vmatrix},$$

that is, differentiating (6.5) is the same as differentiating the last row of (6.5).

The next two lemmas are the generalizations of lemma 5.4 and lemma 5.5 to arbitrary $k \geq 2$.

Lemma 6.3. *Let $f \in C^k([0, 1])$ and let $(f^{(-j)})_{j=1}^k$ be the antiderivatives of f for which,*

(i)

$$f^{(-j)}(1) = \begin{cases} 0, & 1 \leq j \leq k-1, \\ \mu^{2k-1}, & j = k. \end{cases}$$

(ii) Assume further that

$$f(0) = \mu^{-1} \text{ and } f^{(-j)}(0) = 0 \text{ for } 1 \leq j \leq k-1.$$

(iii) For all $x \in [0, 1]$,

$$\Delta_j(f^{(-j)})(x) \neq 0, \text{ for } j = 0, \dots, k-1$$

and

$$\Delta_k(f^{(-k)})(x) = 0$$

Then the set of $4k-2$ functions listed in (6.8), (6.9) and (6.10) below is a solution to (6.4).

$$(6.8) \quad \begin{cases} w_{11} = f \\ w_{22} = w_{2k,2k} = -\frac{1}{\mu} \frac{\Delta_1(f^{(-1)})}{f^2} \\ w_{2,2k} = \frac{1}{\mu^2} \frac{f^{(-1)} \Delta_1(f^{(-1)})}{f^2} \\ w_{2k,2} = \frac{f^{(1)}}{f^2} \end{cases}.$$

For $j = 1, \dots, k-2$

$$(6.9) \quad \begin{cases} w_{j+2,j+2} = w_{2k-j,2k-j} = -\frac{1}{\mu} \frac{\Delta_{j-1}(f^{(1-j)}) \Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\ w_{j+2,2k-j} = \frac{1}{\mu^{2j+2}} \frac{\Delta_j(f^{(-1-j)}) \Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\ w_{2k-j,j+2} = \mu^{2j} \frac{\Delta_{j-1}(f^{(1-j)}) \Delta_j(f^{(1-j)})}{\Delta_j(f^{(-j)})^2} \end{cases}.$$

$$(6.10) \quad w_{k+1,k+1} = \mu^{2k+2} \frac{\Delta_{k-2}(f^{(2-k)}) \Delta_{k-1}(f^{(2-k)})}{\Delta_{k-1}(f^{(1-k)})^2}$$

Moreover for $j = 0, \dots, k-2$

$$(6.11) \quad \begin{vmatrix} w_{j+2,j+2} & w_{j+2,2k-2} \\ w_{2k-j,j+2} & w_{2k-j,2k-j} \end{vmatrix} = \frac{1}{\mu} w_{j+2,j+2}$$

and

$$(6.12) \quad \begin{cases} L(w_{11}) = -f^{(-1)} \\ L(w_{j+2,2k-j}) = -\frac{1}{\mu^{2j+2}} \frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}, \quad 0 \leq j \leq k-3 \\ L(w_{k,k+2}) = \mu - \frac{1}{\mu^{2k-2}} \frac{\Delta_{k-1}(f^{(-k)})}{\Delta_{k-2}(f^{(2-k)})} \end{cases}$$

$$(6.13) \quad \begin{cases} L^*(w_{2k,2}) = \mu - \frac{1}{f} \\ L^*(w_{2k-j,2+j}) = -\mu^{2j} \frac{\Delta_{j-1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}, \quad 1 \leq j \leq k-2. \\ L^*(w_{k+1,k+1}) = -\mu^{2k-2} \frac{\Delta_{k-2}(f^{(3-k)})}{\Delta_{k-1}(f^{(1-k)})} \end{cases}$$

Proof. Let $w_{11}, w_{22}, \dots, w_{kk}, w_{2,2k}, w_{3,2k-1}, \dots, w_{2k,2}$ be given by (6.8), (6.9) and (6.10). Then for $1 \leq j \leq k-2$ the left hand side of (6.11) is equal to

$$-\frac{1}{\mu^2} \frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)})A}{\Delta_j(f^{(-j)})^4},$$

where $A = \Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)}) - \Delta_j(f^{(1-j)})\Delta_j(f^{(-1-j)})$.

By applying (6.6) to $g = f^{(-1-j)}$ it follows that $A = -\Delta_j(f^{(-j)})^2$, which proves (6.11) for $1 \leq j \leq k-2$. The case $j = 0$ of (6.11) follows immediately from (6.8).

The proofs of (6.12) and 6.13) can be obtained exactly as in the case $k = 2$ provided the following two identities holds: For $j = 0, \dots, k-2$:

$$(6.14) \quad \frac{d}{dx} \left(\frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})} \right) = \frac{\Delta_j(f^{(-1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2}$$

For $j = 1, \dots, k-1$:

$$(6.15) \quad \frac{d}{dx} \left(\frac{\Delta_{j-1}(f^{(2-j)})}{\Delta_j(f^{(-j)})} \right) = \frac{\Delta_{j-1}(f^{(1-j)})\Delta_j(f^{(1-j)})}{\Delta_j(f^{(-j)})^2}$$

However (6.14) follows from (6.7) with $g = f^{(-2-j)}$ after changing j in (6.7) to $j+1$. In the same way (6.15) follows from (6.7) with $g = f^{(-j)}$ and j unchanged. It remains to be proved, that $w_{11}, \dots, w_{kk}, w_{2,2k}, \dots, w_{2k,2}$ form a solution to (6.4). The proof of the first 2 identities in (6.4) is exactly the same as in the case $k = 2$. Let us check the next $k-2$

identities in (6.4) i.e.

$$(6.16) \quad \begin{pmatrix} 0 & L(w_{j+1,2k+1-j}) \\ L^*(w_{2k-1-j,j+3}) & 0 \end{pmatrix} + \begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathcal{D})}$$

for $j = 1, \dots, k-2$. By (6.11) and the fact that $w_{2+j,2+j} = w_{2k-j,2k-j}$ (cf. (6.8)) we have

$$\begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \begin{pmatrix} \mu 1_{\mathcal{D}} & \beta \\ \gamma & \mu 1_{\mathcal{D}} \end{pmatrix},$$

where

$$\beta = -\mu \frac{w_{2+j,2k-j}}{w_{2+j,2+j}} = \frac{1}{\mu^{2j}} \frac{\Delta_j(f^{(-1-j)})}{\Delta_{j-1}(f^{(1-j)})}$$

and

$$\gamma = -\mu \frac{w_{2k-j,2+j}}{w_{2+j,2+j}} = \mu^{2j+2} \frac{\Delta_j(f^{(1-j)})}{\Delta_{j+1}(f^{(-1-j)})}.$$

Hence by (6.12) and (6.13)

$$\beta = -L(w_{j+1,2k-j+1}) \text{ and } \gamma = -L^*(w_{2k-1-j,j+3})$$

for $j = 1, \dots, k-2$. This proves (6.16). Observe next that by (6.10) and (6.12)

$$\begin{aligned} w_{k+1,k+1}(\mu - L(w_{k,k+2})) &= \frac{\Delta_{k-1}(f^{(2-k)})\Delta_{k-1}(f^{(-k)})}{\Delta_{k-1}(f^{(1-k)})^2} \\ &= 1 + \frac{\sigma}{\Delta_{k-1}(f^{(1-k)})^2}, \end{aligned}$$

where

$$\sigma = \Delta_{k-1}(f^{(2-k)})\Delta_{k-1}(f^{(-k)}) - \Delta_{k-1}(f^{(1-k)})^2.$$

By (6.6) and the assumptions (iii) in lemma 6.3

$$\sigma = \Delta_{k-2}(f^{(2-k)})\Delta_k(f^{(-k)}) = 0.$$

Hence $w_{k+1,k+1}(\mu - L(w_{k,k+2})) = 1$, which proves the last equality in (6.4). This completes the proof of lemma 6.3. \square

Lemma 6.4. *Let $k \in \mathbb{N}, k \geq 2$ and let $\alpha_j(s), \gamma_j(s)$ for $j = 1, \dots, k$ and $0 < |s| < \frac{1}{e}$ be as in lemma 5.2. Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{e}$, put $s = \frac{1}{k}\mu^{-2}$ and*

$$(6.17) \quad \begin{cases} f(x) = \frac{1}{\mu} \left(\sum_{\nu=1}^k \gamma_{\nu}(s) e^{k\alpha_{\nu}(s)x} \right), & x \in \mathbb{R} \\ f^{(-j)}(x) = \frac{1}{\mu k^j} \left(\sum_{\nu=1}^k \frac{\gamma_{\nu}(s)}{\alpha_{\nu}(s)^j} e^{k\alpha_{\nu}(s)x} \right), & x \in \mathbb{R}, j = 1, \dots, k \end{cases}.$$

Then

(i) $(f^{(-j)})_{j=1}^k$ are successive antiderivatives of f . Moreover

$$(6.18) \quad \begin{cases} f^{(-j)}(1) = 0, & 1 \leq j \leq k-1 \\ f^{(-k)}(1) = \mu^{2k-1} \end{cases}$$

and

$$(6.19) \quad \begin{cases} f(0) = \mu^{-1} \\ f^{(j)}(0) = 0, & 1 \leq j \leq k-1 \end{cases}.$$

(ii) The following asymptotic formulas holds for $|\mu| \rightarrow \infty$

$$(6.20) \quad \begin{cases} f^{(-k)}(x) = \mu^{2k-1} + \mathcal{O}(\mu^{-1}) \\ f^{(-j)}(x) = \frac{1}{j!}(x-1)^j \mu^{-1} + \mathcal{O}(\mu^{-2k-1}), & 1 \leq j \leq k-1 \\ f(x) = \mu^{-1} + \mathcal{O}(\mu^{-2k-1}) \\ f^{(j)}(x) = \frac{1}{j!}x^j \mu^{-2k-1} + \mathcal{O}(\mu^{-4k-1}), & 1 \leq j \leq k-1 \\ f^{(k)}(x) = \mu^{-2k-1} + \mathcal{O}(\mu^{-4k-1}) \end{cases},$$

where the error estimates holds uniformly in x on compact subsets of \mathbb{R} .

(iii) There exists a $\mu_0 \geq \sqrt{e}$, such that the restriction of f to $[0, 1]$ satisfies all the conditions in lemma 6.3, when $|\mu| > \mu_0$.

Proof. From the proof of [DH2, Prop. 4.2], we know that $\alpha_j(s)$ and $\gamma_j(s)$ are analytic functions of $s \in B(0, \frac{1}{e})$. Moreover by [DH1, Prop. 4.1]

$$(6.21) \quad \begin{cases} \sum_{\nu=1}^k \gamma_\nu(s) = 1 \\ \sum_{\nu=1}^k \gamma_\nu(s) \alpha_\nu(s)^j = 1, & j = 1, \dots, k-1 \end{cases}.$$

Moreover, since $\alpha_j(s) = \rho(e^{i\frac{2\pi j}{k}}s)$, where ρ satisfies

$$\rho(w)e^{-\rho(w)} = w \text{ for } |w| < \frac{1}{e}$$

we have $(\alpha_\nu(s)e^{-\alpha_\nu(s)})^k = s^k$ and therefore

$$(6.22) \quad e^{k\alpha_\nu(s)} = \frac{s^k}{(\alpha_\nu(s))^k}$$

for $\nu = 1, \dots, k$. Having (6.21) and (6.22) in mind, the proof of (i) and (ii) in lemma 6.4 is now a routine generalization of the proof of lemma

5.5. Concerning (iii) in lemma 6.4, we have

$$(6.23) \quad \begin{cases} \Delta_j(f^{(-j)}) = \sigma(j)\mu^{-j-1} + \mathcal{O}(\mu^{-2k-j-1}), & 0, \dots, k-1, \\ \text{where } \sigma(j) = 1 \text{ for } j = 0, 3 \pmod{4} \\ \text{and } \sigma(j) = -1 \text{ for } j = 1, 2 \pmod{4} \end{cases}$$

because the leading term in the determinant $\Delta_j(f^{(-j)})$ comes from the antidiagonal, i.e.

$$\Delta_j(f^{(-j)}) = \begin{vmatrix} 0 & \dots & 0 & f \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots \\ f & 0 & \dots & 0 \end{vmatrix} + \mathcal{O}(\mu^{-2k-j-1}) = \sigma(j)f^{j+1} + \mathcal{O}(\mu^{-2k-j-1})$$

since the matrix in question has size $j+1$. Hence $\Delta_j(f^{(-j)})(x) \neq 0$ for $x \in [0, 1]$ and $0 \leq j \leq k-1$, when $|\mu|$ is sufficiently large. Moreover $\Delta_k(f^{(-k)}) = 0$ for $x \in [0, 1]$, because in analogy with (5.29), $\Delta_k(f^{(-k)}(x))$ is the determinant of the $(k+1) \times (k+1)$ matrix

$$F = (f^{(i+j-k)})_{i,j=0,\dots,k}$$

which has the factorization $F = ADA^t$, where A is the $(k+1) \times k$ matrix with entries

$$a_{il} = (k\alpha_l(s))^i, \quad i = 0, \dots, k, \quad l = 1, \dots, k$$

and D is the $k \times k$ diagonal matrix, with diagonal entries

$$d_{ll} = \frac{\gamma_l(s)}{(k\alpha_l(s))^k} e^{k\alpha_l(s)}, \quad l = 1, \dots, k.$$

□

Proof of Theorem 5.1 in the general case. Let μ_0 be as in lemma 6.4, let $\mu \in \mathbb{C}$, $|\mu| > \mu_0$ and put $s = \frac{1}{k}\mu^{-2}$. Put as before

$$f(x) = \frac{1}{\mu} \left(\sum_{\nu=1}^k \gamma_\nu(s) e^{k\alpha_\nu(s)x} \right)$$

for $x \in [0, 1]$, and define $w_{11}, w_{22}, \dots, w_{k,k}, w_{2,2k}, w_{3,2k-1}, \dots, w_{2k,2}$ by (6.8), (6.9) and (6.10), and put all other entries of $w \in M_{2k}(\mathcal{D})$ equal to 0. Then by lemma 6.4, (6.4) holds, and therefore

$$\mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_{2k}(\mathcal{D})}.$$

Let $\delta > 0$ be chosen according to lemma 2.1. If we can find a $\mu_1 \geq \max\{\mu_0, \frac{1}{\delta}\}$, such that

$$(6.24) \quad |\mu| \geq \mu_1 \Rightarrow \|w\| < \delta$$

then $w = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1})$. In particular

$$(6.25) \quad f = w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k}(T^*)^k T^k)^{-1}),$$

and the proof of theorem 5.1 for $k \geq 2$ can be completed exactly as in the case $k = 2$. By (6.23)

$$(6.26) \quad \begin{cases} \Delta_j(f^{(-j)}) = \mathcal{O}(\mu^{-j-1}), & 0 \leq j \leq k-1 \\ \frac{1}{\Delta_j(f^{(-j)})} = \mathcal{O}(\mu^{j+1}), & 0 \leq j \leq k-1 \end{cases}$$

uniformly in $x \in [0, 1]$ for $|\mu| \rightarrow \infty$. We claim that

$$(6.27) \quad \begin{cases} \Delta_j(f^{(-j-1)}) = \mathcal{O}(\mu^{-j-1}), & 0 \leq j \leq k-2 \\ \Delta_{k-1}(f^{(-k)}) = \mathcal{O}(\mu^k) \\ \Delta_j(f^{(1-j)}) = \mathcal{O}(\mu^{-j-2k-1}), & 0 \leq j \leq k-2 \\ \Delta_{k-1}(f^{(2-k)}) = \mathcal{O}(\mu^{-3k}) \end{cases}.$$

Recall by definition 6.1 that

$$\Delta_j(g) = \det((g^{(k+l)})_{k,l=0,\dots,j}).$$

Hence for $0 \leq j \leq k-2$, $\Delta_j(f^{(-j-1)})$ is the determinant of a $(j+1) \times (j+1)$ matrix, where each entry is equal to one of the functions $f^{(-j-1)}, f^{(-j)}, \dots, f^{(j-1)}$. By (6.20) all these functions are of order $\mathcal{O}(\mu^{-1})$ as $|\mu| \rightarrow \infty$. Hence $\Delta_j(f^{(-j-1)}) = \mathcal{O}(\mu^{-j-1})$ proving the first estimate in (6.27). By the same argument, $\Delta_{k-1}(f^{(-k)})$ is the determinant of a $k \times k$ matrix for which the upper left entry is of the order $\mathcal{O}(\mu^{2k-1})$ and all the other entries are of order $\mathcal{O}(\mu^{-1})$. Hence $\Delta_{k-1}(f^{(-k)}) = \mathcal{O}(\mu^{2k-1}(\mu^{-1})^{k-1}) = \mathcal{O}(\mu^k)$. Let $0 \leq j \leq k-1$. Then $\Delta_j(f^{(1-j)})$ is by (6.20) a determinant of a $(j+1) \times (j+1)$ matrix $M = (m_{k,l})_{k,l=0,\dots,j}$ for which

$$\begin{cases} m_{k,l} = \mathcal{O}(\mu^{-1}) & \text{when } k+l < 0 \\ m_{k,l} = \mathcal{O}(\mu^{-2k-1}) & \text{when } k+l \geq 0 \end{cases}.$$

Hence for any permutation π of $\{0, 1, \dots, k\}$ the product

$$m_{0\pi(0)} m_{1\pi(1)} \cdots m_{j\pi(j)}$$

contains at least one factor of order $\mathcal{O}(\mu^{-2k-1})$. Therefore

$$\Delta_j(f^{(1-j)}) = \det(M) = \sum_{\pi \in S_{j+1}} (-1)^{\text{sign}(\pi)} m_{0\pi(0)} m_{1\pi(1)} \cdots m_{k\pi(k)}$$

is of order $\mathcal{O}(\mu^{-2k-1}(\mu^{-1})^j) = \mathcal{O}(\mu^{-2k-j-1})$. This proves the last two estimates in (6.27). Clearly all estimates holds uniformly in $x \in [0, 1]$.

Combining (6.8), (6.9), (6.10) and (6.27), we get

$$\begin{cases} w_{l,l} = \mathcal{O}(\mu^{-1}), & 1 \leq l \leq 2k \\ w_{j+2,2k-j} = \mathcal{O}(\mu^{-2j-3}), & 0 \leq j \leq k-2 \\ w_{2k-j,j+2} = \mathcal{O}(\mu^{2j+1-2k}), & 0 \leq j \leq k-2 \end{cases}.$$

In particular all the entries of w are of size $\mathcal{O}(\mu^{-1})$ as $|\mu| \rightarrow \infty$ uniformly in $x \in [0, 1]$. Hence there exists $\mu_1 \geq \max\{\mu_0, \frac{1}{\delta}\}$ such that (6.24) holds. Hence by (6.25) we have for $|s| < \frac{1}{k}\mu_1^{-2}$,

$$\sum_{k=0}^{\infty} (ks)^{nk} E_{\mathcal{D}}(((T^*)^k T^k)^n)(x) = \sum_{\nu=1}^{\infty} \gamma_j(s) e^{k\alpha_j(s)x}, \quad x \in [0, 1].$$

Now Theorem 5.1 follows from lemma 5.2 and [DH2, remark 4.3] as in the case $k = 2$. \square

APPENDIX A. DETERMINANT-IDENTITIES ON HANKEL-MATRICES

We need the following lemma on Hankel-determinants.

Lemma A.1. *Let $a_{-(n-1)}, a_{-(n-2)}, \dots, a_{n-1}, a_n \in \mathbb{C}$ for some $n \in \mathbb{N}$. Then*

$$(a) \quad \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & \ddots & a_0 \\ a_{-(n-3)} & a_{-(n-4)} & \ddots & a_0 \\ a_{-(n-4)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-4} \\ a_0 & \ddots & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & a_{-(n-3)} & a_{-(n-4)} & \ddots & a_0 \\ a_{-(n-2)} & a_{-(n-3)} & a_{-(n-4)} & \ddots & \ddots & \ddots \\ a_{-(n-3)} & a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} \\ a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} & a_{n-3} \\ \ddots & \ddots & \ddots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_0 & \ddots & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix} \\ = \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & \ddots & a_{-1} \\ a_{-(n-2)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-4} \\ a_{-1} & \ddots & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \ddots & a_1 \\ a_{-(n-4)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-2} \\ a_1 & \ddots & a_{n-2} & a_{n-1} \end{vmatrix} - \begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & \ddots & a_0 \\ a_{-(n-3)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-3} \\ a_0 & \ddots & a_{n-3} & a_{n-2} \end{vmatrix}^2.$$

(b)

$$\begin{aligned}
& \begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & \ddots & a_1 \\ a_{-(n-3)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-1} \\ a_1 & \ddots & a_{n-1} & a_n \end{vmatrix} \begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & \ddots & a_0 \\ a_{-(n-3)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-3} \\ a_0 & \ddots & a_{n-3} & a_{n-2} \end{vmatrix} \\
&= \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & \ddots & a_0 \\ a_{-(n-2)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-3} \\ a_{-1} & \cdots & a_{n-3} & a_{n-2} \\ a_1 & a_2 & \cdots & a_n \end{vmatrix} \begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \ddots & a_1 \\ a_{-(n-4)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-2} \\ a_1 & \ddots & a_{n-2} & a_{n-1} \end{vmatrix} \\
&- \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & \ddots & a_0 \\ a_{-(n-2)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-2} \\ a_0 & \ddots & a_{n-2} & a_{n-1} \end{vmatrix} \begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \ddots & a_1 \\ a_{-(n-4)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{n-3} \\ a_2 & a_3 & \cdots & a_n \end{vmatrix}.
\end{aligned}$$

Proof. To prove (a) we actually prove the more general equation

$$\begin{aligned}
& (A.1) \quad \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \\
&- \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} \end{vmatrix}
\end{aligned}$$

for $a_{ij} \in \mathbb{C}$ and $i, j \in \{1, \dots, n\}$.

We first add some zero terms to the left-hand side (LHS) of (A.1).

$$\begin{aligned}
& \text{LHS} = \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{11} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix} \\
&+ \sum_{k=2}^{n-1} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ a_{31} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,k} & a_{1,k} & a_{1,k+1} & \cdots & a_{2,n-1} \\ a_{22} & \cdots & a_{2,k} & a_{2,k} & a_{2,k+1} & \cdots & a_{3,n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,k} & a_{n,k} & a_{n,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix}
\end{aligned}$$

We note that the last matrix in the sum is zero because column $k-1$ and k are equal. Now we expand LHS after the k 'th column of the

second matrix in the k 'th addent. We get

$$\begin{aligned} \text{LHS} &= \sum_{j=1}^n (-1)^{1+j} a_{j,1} \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \\ &+ \sum_{k=2}^{n-1} \sum_{j=1}^n (-1)^{k+j} a_{j,k} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \end{aligned}$$

where $j = 1$ and $j = n$ means leave out row 1 and n respectively. Switching the indices we have

$$\begin{aligned} \text{(A.2) LHS} &= \sum_{j=1}^n \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \left((-1)^{1+j} a_{j,1} \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \right. \\ &\quad \left. \sum_{k=2}^{n-1} (-1)^{k+j} a_{j,k} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \right) \end{aligned}$$

But the parenthesis on the right-hand side is exactly expansion along the j 'th row of the following determinants

$$\text{(A.3) } \begin{cases} \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}, & j = 1 \\ \begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{j,1} & \cdots & a_{j,n-1} \\ a_{j+1,1} & \cdots & a_{j+1,n-1} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix} = 0, & 2 \leq j \leq n-1 \\ - \begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix}, & j = n. \end{cases}$$

Combining (A.2) and (A.3) we obtain the right-hand side of (A.1) and thus also the proof of (a).

To prove (b) we prove the more general equation

$$\begin{aligned}
 (A.4) \quad & \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2,n} \\ a_{31} & a_{32} & \cdots & a_{3,n} \\ a_{41} & a_{42} & \cdots & a_{4,n} \\ \vdots & \vdots & & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix} \\
 &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \\
 &\quad - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n} \end{vmatrix}
 \end{aligned}$$

for $a_{ij} \in \mathbb{C}$, $i \in \{1, \dots, n+1\}$ and $j \in \{1, \dots, n\}$. We remark that Hankel-matrices are symmetric and for these (A.4) reduces to (b). Observe that for $k \in \{2, \dots, n\}$ we have

$$\begin{aligned}
 0 &= (-1)^k \begin{vmatrix} a_{1,k} & a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{2,k} & a_{12} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,k} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \\ a_{n+1,k} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n} \end{vmatrix} \\
 &= (-1)^k \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix},
 \end{aligned}$$

where the $j = 1$ and $j = n+1$ are interpreted as remove the 1st and $(n+1)$ th coloumn respectively. Thus also

$$\begin{aligned}
 0 &= \sum_{k=2}^n \begin{vmatrix} a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n} \\ a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n} \end{vmatrix} \\
 &\quad \cdot \left((-1)^k \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \right)
 \end{aligned}$$

Switching the indices we have

$$(A.5) \quad 0 = \sum_{j=1}^{n+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \cdot \left(\sum_{k=2}^n (-1)^{k+j-1} a_{j,k} \begin{vmatrix} a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n} \\ a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n} \end{vmatrix} \right)$$

The parenthesis of (A.5) is expansion along the j^{th} row of the following expression except for $j = n + 1$ where we expand along the n^{th} row.

$$(A.6) \quad \begin{cases} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix}, & j = 1 \\ 0, & j \in \{2, \dots, n-1\} \\ \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix}, & j = n \\ - \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n} \end{vmatrix} & j = n+1. \end{cases}$$

Combining (A.5) and (A.6) we obtain (A.4) and this finishes the proof of (b). \square

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